Lecture 2.

End of last time introduced partition function
in presence external source J

$$Z(J) = \int D\phi \, e^{-S(J)/\hbar}$$

(Euclidean space)

Define functional integral: introduced notion of
a lattice: lattice spacing

$$\varphi(x) \to \phi_i \quad i = 1 \ldots L^4 \text{ say} \quad a$$

$$\Rightarrow \; Z(J) = \int \prod_i d\phi_i \, e^{-S(\phi, J)}$$

$\boxed{\text{set } \hbar = 1}$

Greens functions

obtained by differentiating wrt J.

all content of theory — LSZ reduction tells you
that all elements of S matrix
obtained from these.

limit $a \to 0$, $L \to \infty$ ~~[illegible]~~ subtle
involves renormalization

In more detail

$$Z(J) = \int_{-\infty}^{\infty} d\phi_1 \cdots \int_{-\infty}^{\infty} d\phi_N \, e^{-\frac{1}{2}\phi^T A \phi - \frac{\lambda}{4!}\phi^4 + J^T\phi}$$

where $\phi^4 = \sum_i \phi_i^4$ set $a=1$ here on.

$$\phi^T A \phi = \sum_{ij} \phi_i A_{ij} \phi_j$$

all terms quadratic in field.

$\left(\int d^4x \to \sum_i a^4 \quad a = \text{lattice spacing} \right)$

If I could evaluate $Z(J)$ exactly could "solve" QFT fully

$\longrightarrow$ predict all Green functions

In practice this can only be done for $\lambda = 0$ free field theory.

Lets see how to do this ...

$$-\tfrac{1}{2}\phi^T A \phi + J^T \phi$$

$$= -\tfrac{1}{2}(\phi - A^{-1}J)^T A(\phi - A^{-1}J) + \tfrac{1}{2} J^T A^{-1} J$$

(assume $A = A^T$)

Change variables $\phi' = \phi - A^{-1}J$.

$\prod d\phi_i = \prod d\phi'_i$ measure invariant

$$\therefore Z(J,\lambda=0) = \underbrace{\int \prod d\phi' \, e^{-\frac{1}{2}\phi' A \phi'}}_{\text{generalized Gaussian integral}} e^{\frac{1}{2}J^T A^{-1} J}$$

$$\left(\frac{(2\pi)^N}{\det A}\right)^{1/2}$$

indep of J.

$$\therefore \langle \phi_i \rangle = \frac{1}{Z}\frac{\delta}{\delta J_i} Z[J]\Big|_{J=0} = 0$$

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z}\frac{\delta^2}{\delta J_i \delta J_j} Z = A^{-1}_{ij}$$

propagator.

diagrammatically $i \bullet \!\!-\!\!\!-\!\!\!-\!\! \bullet j$ amplitude to propagate $i \to j$ free theory.

Continuum expression propagator A^{-1}?

(Eucl. space)

$$A \to -\partial^2 + m^2$$

$$AA^{-1} = I \to (-\partial^2+m^2) D(x-y) = \delta^4(x-y)$$

$\uparrow$ $\frac{\delta_{xy}}{a^4}$

ie $D(x-y) \sim A^{-1}_{ij}$

depends only on difference $x-y$ by translation invariance

$$J^T A^{-1} J \to \int\int d^4x\, d^4y\, J(x) D(x-y) J(y)$$

writing $\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}$

$$\to (k^2+m^2) D(k) = 1$$

or $D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{(k^2+m^2)}$ ← note pole at $k = \pm im$ ↳ $e^{-k(x-y)}$ behavior

Minkowski space ?

$m^2 \to m^2 - i\epsilon$ so that integrand contour $e^{-\epsilon\int d^4x\,\phi^2}$ convergence

$$\to D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon}$$

(1)

$$D(x) = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{e^{ik_0x_0} e^{ik\cdot x}}{k_0^2 - k^2 - m^2 + i\epsilon}$$

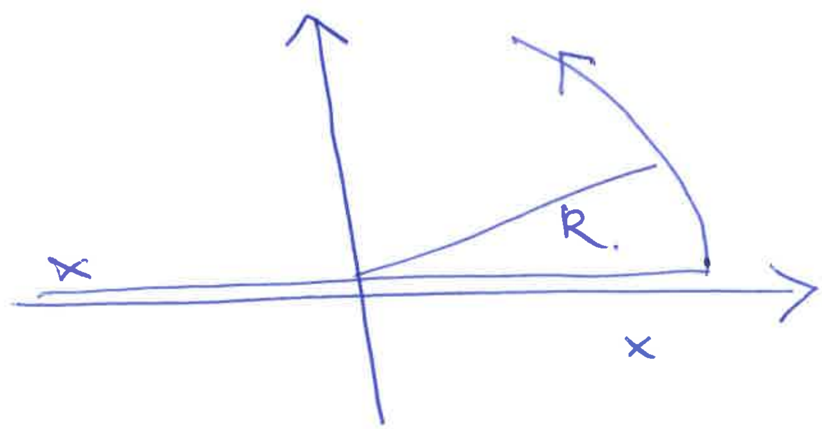
poles at $\omega_k - i\epsilon$ & $-\omega_k + i\epsilon$ $\omega_k = \sqrt{k^2+m^2}$

evaluate as contour integral

complete contour in upper/lower half plane according to sign of x_0

if $x_0 > 0$ complete in upper so $e^{ik_0x_0} \to e^{-|k_0|R}$

yields zero $R \to \infty$



pole at $-\omega_k + i\epsilon$ has residue

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i(-\omega_k t - kx)}}{-2\omega_k}$$

$\therefore$ integral $= -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i(-\omega_k t - kx)}$

if $x_0 < 0$ close in lower ½ plane picking up other pole.

↳ integral $-i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i(\omega_k t - kx)}$

thus

$$D(x) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{i(-\omega_k t + kx)} \theta(t) + e^{i(\omega_k t - kx)} \theta(-t) \right]$$

lets examine causality structure of this propagator.

Assume x_μ is timelike. This means I can find a FOR where $\vec{x} = 0$

$$\hookrightarrow D(x) \sim \int \frac{d^3k}{\omega_k} e^{-i\omega_k t} \quad \text{if } t > 0$$

However if x spacelike $x^0 = 0$ $(t=0)$

$$D(x) \sim \int \frac{d^3k}{\omega_k} e^{-ik \cdot x}$$

do k integral. dominated by poles at $k = \pm im$.

$$\sim e^{-m|x|}$$

→ quantum particles can "leak out" of lightcone distance m^{-1} (Heisenberg)

Summarize

Scalar field describes spin 0 particles

Lorentz invariant

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \lambda/4!\,\phi^4 + \ldots$$

Quantum theory defined by path integral (+source)

$$Z(J) = \int D\phi\, e^{iS/\hbar}$$

In practice — Euclidean space $t = -i\tau$

& define $D\phi$ on lattice

$$\hookrightarrow \quad Z(J) = \int \prod d\phi\, e^{-S(\phi, J)}$$

well-defined object

Green functions by differentiation wrt J

~~Pa~~ Most important of these is 2p function

$$G^2(x_1, x_2) = \langle \phi(x_1)\,\phi(x_2) \rangle$$

propagator

inverse of quadratic form in S

describes propagation of free particles

encapsulates correct causal structure

move on to $\lambda \neq 0$ ...

O.K back to

$Z(J,\lambda)$ (Eucl. space)

on lattice

power expand in case of interactions..

$$Z(J,\lambda) = \int \prod d\phi \, e^{-\frac{1}{2}\varphi^T A \varphi + J\varphi} \times$$

$$\left[1 - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \varphi^8 + \dots\right]$$

rewriting $\Rightarrow$

$$Z(J,\lambda) = \left(1 - \frac{\lambda}{4!}\sum_i \frac{\delta^4}{\delta J_i^4} + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \sum_{ij} \frac{\delta^8}{\delta J_i \dots \delta J_j} + \dots\right)$$

ie $\times \; Z(J,0)$

$$\boxed{Z(J,\lambda) = e^{-\lambda/4! \sum_i \frac{\delta^4}{\delta J_i^4}} Z(J,0)}$$

$$= C e^{-\lambda/4! \sum \frac{\delta^4}{\delta J_i^4}} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$C = \left(\frac{(2\pi)^N}{\det A}\right)^{1/2}$$

yields perturbative expansion of Z in λ.

Quick reminder how this all looks in continuum (Eucl. space)

$$Z(J,\lambda) = e^{-\lambda/4! \int d^4w \frac{\delta^4}{\delta J(w)^4}} Z(J,0)$$

when

$$Z(J,0) = \frac{C}{\det^{1/2}(\partial_\mu \partial_\mu + m^2)} e^{\frac{1}{2}\int d^4x\, d^4y\, J(x) D(x-y) J(y)}$$

ill-defined but no matter for now!

Green's function expansion

Consider Taylor expanding $Z(\lambda, J)$ in powers of J

$$Z(\lambda, J) = \sum_{s=0}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_s=1}^{N} \frac{1}{s!} J_{i_1} \cdots J_{i_s} G^{(s)}_{i_1 \cdots i_s}$$

remember

$$G^{(s)}_{i_1 \cdots i_s} = \frac{\delta^s}{\delta J_1 \cdots \delta J_{i_s}} Z(J, \lambda)$$

$$= \int \prod d\phi \, \phi_{i_1} \cdots \phi_{i_s} e^{-S(\phi)}$$

#

reminder: in cont^m

$$Z(\lambda, J) = \sum_{s=0}^{\infty} \frac{1}{s!} \int d^4x_1 \cdots d^4x_s \, J(x_1) \cdots J(x_s) \, G^{s}(x_1 \cdots x_s)$$

functional power series

$$\left. \frac{\delta^s Z}{\delta J(x_1) \cdots \delta J(x_s)} \right|_{J=0}$$