Lecture 3

(14.5)

Recap

* Scalar Field theory
* Partition function determines

Green functions G^n (correlation functions)

coeffs in functional Taylor expansion

of $Z(J)$ in powers of J.

$G^n \to$ scattering amplitudes (S matrix)

also related to effective quantum action

$Z(J)$ can only be computed exactly

in free theory

(lattice, Eucl. space)

$$Z(J,0) \sim e^{JA^{-1}J}$$

$A^{-1} \to$ propagator — amplitude ~~to~~ for particle to travel

$i \to j$

Perturbation theory

O.K, lets bite the bullet and start computing Greens functions

free theory $\lambda = 0$

$$G^4(1,2,3,4) = \frac{\partial^4}{\partial J_1 \ldots \partial J_4} Z(J,\lambda)\Big|_{J=0}$$

$$\& \quad Z(J,0) = C e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

lattice model

Each pair of $\delta/\delta J$'s brings down factor of A^{-1}_{ij}

Since set $J=0$ at end each term you get must take form

$$A^{-1}_{12} A^{-1}_{34}$$ (all odd derivs vanish)

but will also get

$$A^{-1}_{13} A^{-1}_{24} + A^{-1}_{14} A^{-1}_{23}$$

ie

in detail

(15.J)

$$\boxed{A = A^T}$$

$$G^4(1,2,3,4) = \frac{\partial^4}{\partial J_1 \partial J_2 \partial J_3 \partial J_4} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$= \frac{\partial^3}{\partial J_1 \partial J_2 \partial J_3} \left(A^{-1}_{4j} J_j e^{[\;]} \right)$$

$$= \frac{\partial^2}{\partial J_1 \partial J_2} \left(A^{-1}_{43} + A^{-1}_{4j} J_j A^{-1}_{3k} J_k \right) e^{[\;]}$$

$$= \frac{\partial}{\partial J_1} \Big(A^{-1}_{43} A^{-1}_{2l} J_l + A^{-1}_{42} A^{-1}_{3k} J_k$$
$$+ A^{-1}_{4j} J_j A^{-1}_{32}$$
$$+ A^{-1}_{4j} J_j A^{-1}_{3k} J_k A^{-1}_{2l} J_l \Big) e^{[\;]}$$

$$= \Big(A^{-1}_{43} A^{-1}_{21} + A^{-1}_{42} A^{-1}_{31}$$
$$+ A^{-1}_{41} A^{-1}_{32} + \underset{\text{at } J=0}{\text{stuff that} \to 0} \Big) e^{[\;]}$$

$$= A^{-1}_{43} A^{-1}_{21} + A^{-1}_{42} A^{-1}_{31} + A^{-1}_{41} A^{-1}_{32}$$

Thus,

$$G^4(1234) = A^{-1}_{13}A^{-1}_{24} + A^{-1}_{14}A^{-1}_{23} + A^{-1}_{12}A^{-1}_{34}$$

$$= \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle + \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle$$

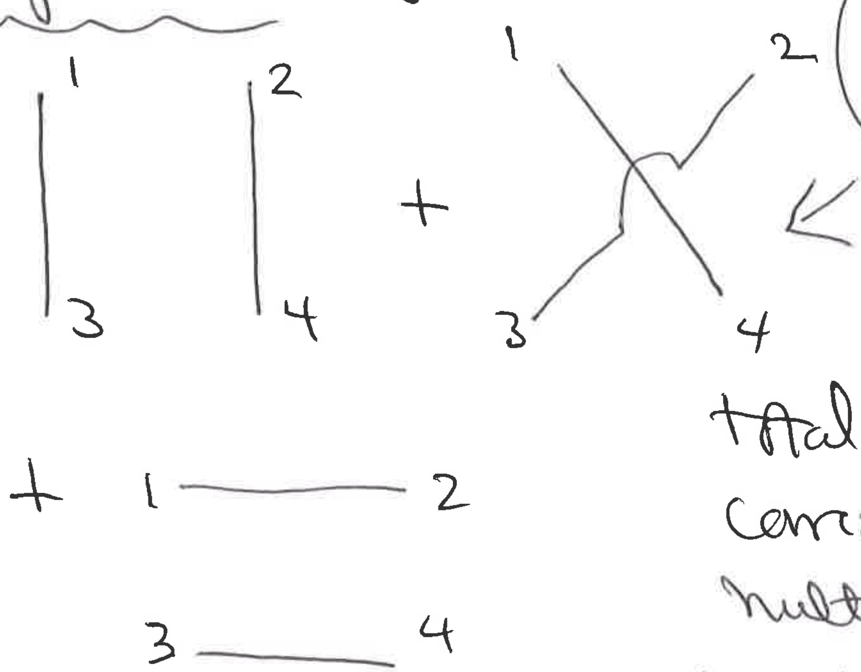
Wick's theorem: sum over all pairwise contractions of fields in n pt function:

trivial consequence of Gaussian integration!

even for $\lambda \neq 0$

Diagrammatically:

Feynman diagrams



total amplitude corresponds to multiplying together amplitudes for pairs of particles to propagate

What about G^4 at $O(\lambda)$?

$$G^4 = \frac{\partial^4}{\partial J_1 \cdots \partial J_4} Z(J,\lambda)$$

$\uparrow$ at $O(\lambda)$

$$= \left(\frac{-\lambda}{4!} \sum_k \frac{\delta^4}{\delta J_k{}^4} e^{\frac{1}{2} J A^{-1} J} \right) \Bigg|_{J=0}$$

Again, each pair of $\frac{\delta}{\delta J\text{'s}}$ brings down factor of A^{-1}

8 J's at $O(\lambda) \longrightarrow$ 4 factors of A^{-1}

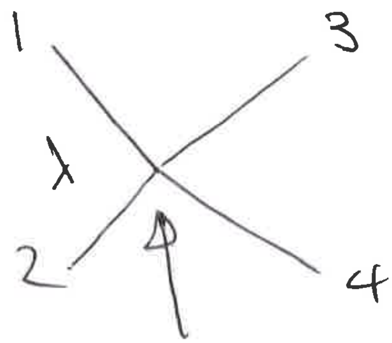
Once one sets $J=0$ at end only surviving terms correspond to all possible ways of distributing labels 1, 2, 3, 4 + 4 k's over these 4 factors of A^{-1}

1 such term:

$$-\lambda \sum_k A^{-1}_{1k} A^{-1}_{2k} A^{-1}_{3k} A^{-1}_{4k}$$

↑ note factor of 1/4! cancels 4! ways pair k with 1, 2, 3 etc.

Corresponding diagram

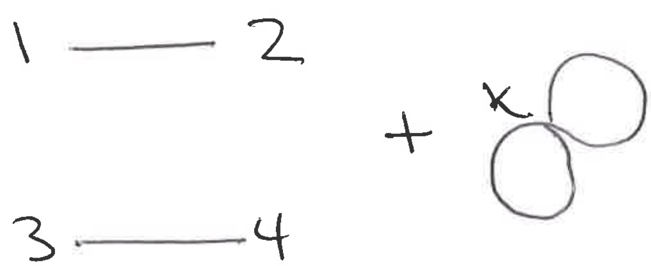


vertex corresponds to ϕ^4 int. term.

$\sum_k$ corresponds to summing over all intermediate positions where the int. occurs

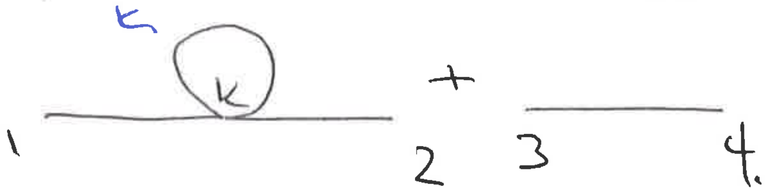
But this is not the only way to arrange the labels

$$\sum_k A^{-1}_{12} A^{-1}_{kk} A^{-1}_{34} A^{-1}_{kk} \quad \times -\lambda$$



← "loops"

also $\sum_k A^{-1}_{1k} A^{-1}_{k2} A^{-1}_{kk} A^{-1}_{34} \cdot -\lambda$



Connected vs disconnected

Notice I can separate all those last 3 diagrams into 2 parts without cutting any lines: they factorize

It turns out diagrams where this cannot be done — so-called connected diagrams are (much) more important than the "disconnected" ones

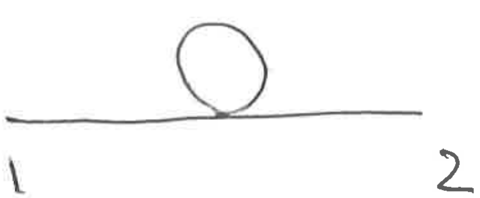
Thus $G^4_c \equiv X$ at $O(\lambda)$ only

4 $-\lambda$ $\langle 1k\rangle\langle 2k\rangle\langle 3k\rangle\langle 4k\rangle$

What about G^2 $O(\lambda)$?

6 $\delta/\delta J$'s $\to$ 3 A^{-1} factors

eg $-\lambda\sum_k A^{-1}_{1k} A^{-1}_{kk} A^{-1}_{k2}$ (note A^{-1} is symmetric)



← connected contribution at $O(\lambda)$ to G^2_c

These pictures are called <u>Feynman diagrams</u>. They summarize the various terms that can arise when computing correlation functions perturbatively in λ

Keep track of terms & give physical picture of what the various terms in the perturbation series correspond to.

Momentum space

Frequently it is convenient to compute G^n in k-space ← to compare with results of scattering expts for example

$$G^4_c(k_1 .. k_4) = \frac{1}{(k_1^2-m^2)}\frac{1}{(k_2^2-m^2)}\frac{1}{(k_3^2-m^2)}\frac{1}{(k_4^2-m^2)} \times \lambda$$

$$\times\, \delta^4(k_1+k_2+k_3+k_4)$$

↑ conservation of k.

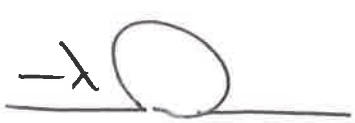
$\langle \phi_1 \phi_2 \rangle$ in k-space.

One wrinkle...

You will have noticed that the "4! included in the definition of the coupling λ is missing from the calculations of G^4 etc.

The reason is easy: it is cancelled by # of permutations of the legs coming out of vertex ← the $\delta/\delta J$'s give you all of these

However, occasionally this overcounts. If the diagram has a certain symmetry there are not 4! different terms corresponding to that diagram

Thus 

$$\to -\frac{1}{2}\lambda \sum_k A^{-1}_{1k} A^{-1}_{kk} A^{-1}_{k2}$$

since when 2 legs connect back they do not yield different diagram

— symmetry factors

Feynman rules

① Draw all topologically distinct Feynman diagrams with V vertices (int^n pts) if working to $O(\lambda^V)$

G^N needs N external propagator lines

② All propagators get factor $\frac{1}{k^2-m^2}$

(for scattering amplitude lop off external propagators – LSZ)

③ Factor of $i\lambda(-\lambda)$ each vertex

④ Conserve 4-momentum at each vertex

⑤ Integrate over closed internal loops

⑥ Divide by symmetry factor S

S reflects # ways of permuting internal vertices and propagators which leave diagram inv^t

Connected / Disconnected Diagrams

Scattering amplitudes related to connected Greens functions / diagrams

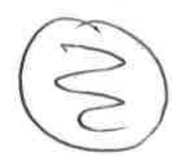
Remarkably

$$Z(J) = e^{-W(J)} \qquad \text{(E S)}$$

$$W(J) = \text{sum of connected diagrams} \qquad (e^{iW} \text{ M S})$$

Why?

Consider $\dfrac{\delta Z}{\delta J} \equiv$  all F.D with 1 external line

$=$  $\times$  sum of all diagrams $= Z$

connected diagrams $\dfrac{\delta W}{\delta J}$

ie $$\frac{\delta Z}{\delta J} = \frac{\delta W}{\delta J} Z$$

$$\longrightarrow \quad Z = e^{W}$$

2d

Thus

$$W(J) = \sum_{s=0}^{\infty} \frac{1}{s!} \frac{\delta}{\delta J_{i_1}} \cdots \frac{\delta}{\delta J_{i_s}} W(J) \Big|_{J=0} J_{i_1} \cdots J_{i_s}$$

$$G_c(i_1 \ldots i_s)$$

connected Green functions

$$W(J) = \ln Z(J)$$

eg) $\langle \phi_i \phi_j \rangle_c = \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \ln Z$

$$= \frac{\partial}{\partial J_i} \frac{1}{Z} \frac{\partial Z}{\partial J_j}$$

$$= \frac{\partial}{\partial J_i} \left(\frac{1}{Z} \int \phi_j e^{-S} \right)$$

$$= \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle$$

note: connected correlators $\to 0$ as args $\to \infty$ — cluster decomposition

since $\langle \phi(x) \phi(y) \rangle \longrightarrow \langle \phi(x) \rangle \langle \phi(y) \rangle$ as $|x-y| \to \infty$

ie $G_c^4 = 0$ automatically in free theory.

Effective Action $\Gamma(\phi)$

$W(J)$ is very useful. But there is another function(al) that is sometimes even more useful. – effective action $\Gamma(\phi)$

Function of field ϕ that gives exact scattering amplitudes of quantum theory using only tree level diagrams!

$$\text{ie } Z[J] = \int D\phi \, e^{i(\Gamma(\phi) + J\phi)}$$

evaluate in limit $\hbar \to 0$
i.e. no loops

$$\therefore \quad \frac{\delta \Gamma}{\delta \phi_c} = -J.$$

$$\text{but } \phi_c = \langle \phi \rangle = \frac{\delta}{\delta J} W \quad *$$

$$\text{or } \quad e^{iW(J)} = e^{i(\Gamma(\phi_c) + J\phi_c)} \quad \leftarrow \int d^4x \, J(x)\, \phi_c(x)$$

$$\to \quad \boxed{\Gamma(\phi_c) = W(J) - J.\phi_c}$$

note J on RHS of this expression is implicitly function of ϕ_c thru eqⁿ $*$

$\Gamma(\phi)$ can be Taylor expanded

$$\Gamma(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1} \cdots \sum_{i_n} \Gamma^{(n)}_{i_1 \cdots i_n} \phi_{i_1} \cdots \phi_{i_n}.$$

proper vertices or 1PI vertices

$$\text{or} \quad \Gamma^{(n)} = \left. \frac{\delta^n \Gamma}{\delta\phi_{c1} \cdots \delta\phi_{cn}} \right|_{\phi_c = 0}$$

$\Gamma^{(n)}$'s determined from ~~E~~ $W(J)$ & hence

<u>connected Greens functions</u> G^n_c

eg/

$$\sum_n \frac{\delta^2 W}{\delta J_1 \delta J_n} \frac{\delta^2 \Gamma}{\delta\phi^c_n \delta\phi_2} = \sum_n \frac{\delta\phi^c_i}{\delta J_n} \frac{\delta J_n}{\delta\phi^c_2}$$

$G^{(2)}_c(1,n)$ $\quad \Gamma^{(2)}_{n2}$ $\quad = \delta_{12}.$

ie $\Gamma^{(2)}_{12} \sim G^{-1}_{12}$

$\leftarrow$ note $\Gamma^{(2)} \sim p^2 + m^2$
$\sim m^2 \quad p^2 \to 0$

ie $\Gamma^{(2)}$ "quantum mass" parameter

In fact can show that

$\Gamma^{(n)}$ are fundamental Green's

functions we need to compute

all other G_c's can be built from them.

They give the (renormalized) coupling

constants of the theory...

Effective Potential

Alternatively can write:

$$\Gamma(\phi_{cl}) = \int \left(-V(\phi_{cl}) + \tfrac{1}{2} Z(\phi_{cl}) (\partial \phi_{cl})^2 + \dots \right)$$

effective potential.

$$\frac{\delta \Gamma}{\delta \phi_c} = \frac{\delta V}{\delta \phi_c} = -J.$$

if ϕ_{cl} indep of position $\quad \delta V / \delta \phi_c = J$

ie ϕ_c solution in ground $\quad = 0$ if J vanishes

state $\leftarrow$ determined by minimizing $V_{eff}(\phi_c)$