Lecture 4

Computation of effective action at one loop.
$\uparrow O(\hbar)$

In steepest descent $\Rightarrow$

$$e^{iW(J)} = \int D\phi\, e^{i(S(\phi)+J\phi)}$$

$$\partial^2\phi_c + V'(\phi_c) = J \qquad \tilde{\phi} + \phi_c = \phi.$$

$$\therefore Z = e^{iW(J)} = \int D\tilde{\phi}\, e^{i(\int \frac{1}{2}(\partial\tilde{\phi})^2 - V''(\phi_c)\tilde{\phi}^2)} \times e^{i(S(\phi_c)+J\phi_c)}$$

expanding to quadratic order
(1 loop)

$$\text{ie } Z(J) \simeq e^{i(S(\phi_c)+J\phi_c) - \frac{1}{2}\mathrm{tr}\ln(\partial^2+V''(\phi_c))}$$

note that here ϕ_c is to be thought of as $f(J)$

$$\text{ie } W(J) = S(\phi_c) + J\phi_c + \frac{i\hbar}{2}\mathrm{tr}\ln(\partial^2+V''(\phi_c)) + O(\hbar^2)$$

$$\therefore \Gamma(\phi) = W(J) - J\phi_c$$

$$\Gamma(\phi) = S(\phi_c) + \frac{i\hbar}{2}\mathrm{tr}\ln(\partial^2+V''(\phi_c))$$

Assume soln ϕ_c indep of x & go to

Eucl. space $\rightarrow$

$$i\,\mathrm{tr}\ln(\partial^2+V'') \rightarrow \int d^4x\, \langle x|\ln(\partial^2+V'')|x\rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4}\langle x|k\rangle\langle k|\ln(\partial^2+V'')|k\rangle\langle k|x\rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4}\ln(k^2+V'')$$

ie $V_{eff}(\phi) = V(\phi) + \frac{\hbar}{2}\int \frac{d^4k}{(2\pi)^4}\ln(k^2+\mu^2(\phi))$

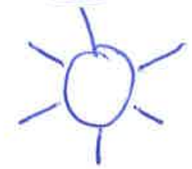
$\mu^2 = m^2 + \frac{1}{2}\lambda\phi^2$

Coleman Weinberg

Suppose $m^2 = 0$

$V(\phi) \sim \lambda/4!\,\phi^4$

classical theory inv under $\phi \rightarrow -\phi$

Σ 

vacuum at $|\phi = 0|$

but quantum mechanically

$V'_{eff}(\phi) = 0$ has

solution away from origin...

$\phi^3 + \frac{\hbar}{2}\phi\int\frac{1}{k^2+\mu^2} = 0$ $\quad\langle\phi\rangle \sim -ve$

spontaneous sym breaking!

CW potential

Expand log

$$\frac{\hbar}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(1+\mu^2/k^2\right)$$

$$= \frac{\hbar}{2}\int \frac{d^4k}{(2\pi)^4}\left(-\mu^2/k^2 + \frac{1}{2}\left(\mu^2/k^2\right)^2 + \dots\right)$$

set $m=0 \to \mu^2 = \frac{1}{2}\lambda\phi^2$

$$\therefore \delta V_{eff} \sim -\frac{1}{2}\lambda\phi^2 \underbrace{A}_{\int d^4k/k^2} + \frac{1}{2}\left(\frac{1}{4}\lambda^2\phi^4\right)\underbrace{B}_{\int d^4k/k^4} + \dots$$

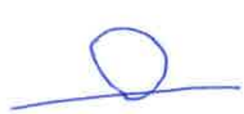
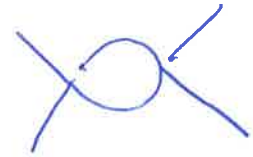
Pain

(1) V_{eff} picks up arbitrary powers of ϕ

i.e. $\Gamma^{(n)} \neq 0$ $n>0$

(2) $m^2 = -$ve $\leftarrow$ min away from origin (spontaneous breaking of $\phi \to -\phi$ sym)

(3) $A, B \to \infty$! problem – later

(4) pictures Γ^2:  Γ^4: 

Spin 1 fields

Symmetries of (flat) spacetime are well known – translations P_μ & Lorentz transformations $\Lambda_{\mu\nu}$ (SO(3,1))

→ make up Poincaré group

$\boxed{\text{Particles transform in irreducible unitary representations of Poincaré}}$

Wigner tells us ~~that~~ these are ① all ∞ dimensional

another key reason for fields

② classified by mass & spin

← reps of SO(3)/SU(2)
$J = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ etc

③ if $J > 0$ & $m \neq 0$
exactly $(2J+1)$ physical state spin J particle.

if $m = 0$ just 2 states

To make local Lorentz invariant field theories we need to know how to embed these unitary reps into objects with spacetime indices ← well-defined reps of Lorentz

this is non-trivial since in general there is a conflict between unitarity & Lorentz invariance

eg suppose I have a 4-vector V_μ.

$$|\psi\rangle = \alpha|V_0\rangle + \beta|V_1\rangle + \gamma|V_2\rangle + \delta|V_3\rangle$$

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 > 0$$

as needed positive norm state.

but this norm is not Lorentz invariant!

requires a different norm

$$|\alpha|^2 - |\beta|^2 - |\gamma|^2 - |\delta|^2$$

but this means $\langle\psi|\psi\rangle$ not pos def!

for spin zero these problems are avoided
since only 1 state

~~[illegible]~~

& $\mathcal{L}(\phi)$ manifestly L.I & propagates
exactly 1 dof ⇐ the correct # for $J=0$

However $J=1$ is more tricky ...

Write down all Lorentz invariant kinetic
terms

$$\mathcal{L} = \frac{a}{2} A_\mu \Box A_\mu + \frac{b}{2} A_\mu \partial_\mu \underbrace{\partial_\nu A_\nu}_{} + \frac{1}{2} m^2 A_\mu^2$$

forces A_μ to be
a 4-vector

But I need to remove 1 dof to agree
with Wigner

ie $4 \to 3 + 1$.

ie A_μ is a reducible rep & need to ensure
that only the 3 propagates - -

EOM $\Longrightarrow$

$$a \Box A_\mu + b \partial_\mu \partial_\nu A_\nu + m^2 A_\mu = 0$$

take ∂_μ of this $\rightarrow$

$$\left[(a+b)\Box + m^2 \right] \partial_\mu A_\mu = 0$$

if $m^2 \neq 0$ & $a = -b$ this requires

$$\boxed{\partial_\mu A_\mu = 0}$$ consequence of EOM

Thus $\mathcal{L}$ with $a=-b$ & $m^2 \neq 0$ actually only propagates 3 dof $\rightarrow$ the spin 0 piece is removed by $\partial \cdot A = 0$ which is implied by EOM.

One can also show that Energy of system ≥ 0 with this choice.

Bounding E below is a necessary condition for unitarity (as is only have up to 2 derivs in $\mathcal{L}$)

But when $a=-b$ I can combine the 2 ~~[illegible]~~ terms

$$\frac{1}{2} A_\mu \Box A_\mu - \frac{1}{2} A_\mu \partial_\mu \partial_\nu A_\nu$$

$$= -\frac{1}{4} F_{\mu\nu}^2 \;!$$ no mention of gauge invariance

~~For~~. $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2$

correct $\mathcal{L}$ ~~that~~ propagate 3 dof.

In general

$$A_\mu(x) = \sum_{i=1}^{3} \int \frac{d^3 p}{(2\pi)^3} a_i(p) \epsilon^i_\mu(p) e^{ipx}.$$

with $p_0 = \sqrt{\vec{p}^2 + m^2}$.

$\partial \cdot A = 0$

$\rightarrow \quad p_\mu \epsilon^i_\mu = 0$

ϵ_μ = polarization vector

3 solutions: 2 transverse

$\epsilon^1 (0,1,0,0)$

$\epsilon^2 (0,0,1,0)$

$\epsilon^3 \sim \left(\frac{p}{m}, 0, 0, E/m\right)$ if $p_\mu = (E,0,0,p)$

What about the massless case?

Naively $\mathcal{L} \to -\frac{1}{4} F_{\mu\nu}^2$.

however notice that if $m^2 = 0$ we lose the automatic constraint $\partial . A = 0$

however the $m=0$ theory has a new property that helps.

$\mathcal{L}$ invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\therefore \partial_\mu A_\mu \to \partial_\mu A_\mu + \Box \alpha$$

can consistently find α such that

$$\partial_\mu . A_\mu = 0$$

eliminating 1 d.o.f.

$\hookrightarrow p_\mu \epsilon_\mu = 0$ 3 possible polarizations

$$\underset{\epsilon_1}{(0,1,0,0)}, \quad \underset{\epsilon_2}{(0,0,1,0)} \quad \& \quad \underset{\uparrow\ \epsilon_L}{(1,0,0,1)}$$

but $\epsilon_L \propto P_\mu$ itself. (37)

so $A_\mu \propto \partial_\mu \Phi$ some ϕ.

but this is pure gauge - not physical.

thus at $m=0$ gauge invariance allows us to restrict to just 2 polarizations in agreement with Wigner.

<u>Moral</u>: gauge invariance is best not thought of as a symmetry. It is a property of the Lagrangian that allows us to propagate the correct # dof to be consistent with unitarity

It allows us to use local, Lorentz invariant field theories to describe massless spin 1 particles using a field A_μ which at first glance has 4 dof....

<u>- reflects</u> underlying redundant description