Lecture 5

quantization of spin 1 (38)

Again, imagine using a path integral.

need source term

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^{\mu}A_{\mu}$$

to ensure that $\partial_\mu A^\mu = 0$ can be maintained

need $\mathcal{L}$ to be G.I

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$\therefore J^\mu \partial_\mu \alpha = 0$ i.e. $\partial_\mu J^\mu = 0$

i.e. couple only to conserved current

Interested in

$$Z(J) = \int DA\, e^{iS} \qquad \text{formally}$$

$$S = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right)$$

MS

but care is needed. You are potentially integrating over ~~equiv~~ gauge copies --

In k space

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{A}_\mu(k)$$

$$\int F_{\mu\nu}F^{\mu\nu} = 2\int \partial_\mu A_\nu F^{\mu\nu} = 2\int \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= 2\int A_\nu(-\Box)A^\nu + A_\nu \partial_\mu \partial^\nu A^\mu)$$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\Big[\tilde{A}_\mu(k)\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\tilde{A}_\nu(-k)\Big] + \tilde{J}^\mu(k)A_\mu(-k) + \tilde{J}^\mu(-k)A_\mu(k)\Big]$$

following along as before want inverse of <u>quadratic operator</u> (photon propagator)

$$M^{\mu\nu} = k^2 g^{\mu\nu} - k^\mu k^\nu$$

zero modes! not invertible!

pure gauge → <u>gauge invariance is to blame</u>.

$$\tilde{A}_\mu(k) \to \tilde{A}_\mu(k) - \underbrace{k_\mu \Phi(k)}_{0 \text{ mode.}}$$

Defining

$$P^{\mu\nu} k^2 = k^2 g^{\mu\nu} - k^\mu k^\nu$$

i.e. $$P^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

$P^{\mu\nu}$ is a <u>projector</u>. That is $P^2 = P$

check

$$\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\left(g_\nu{}^\lambda - \frac{k_\nu k^\lambda}{k^2}\right)$$

$$= g^{\mu\lambda} + \frac{k^\mu k^\lambda}{k^2} - \frac{2 k^\mu k^\lambda}{k^2}$$

$$= P^{\mu\lambda}$$

also $\boxed{P^{\mu\nu} k_\nu = 0}$

<u>Note</u>: any mode $A_\mu(k) \propto k_\mu$ is annihilated by quadratic op / $P \Rightarrow S$ does not depend on such modes $\leftarrow$ pure gauge

$\therefore$ prescription should be to <u>not</u> integrate over them in P.I !

Equivalent to projecting other out of $A_\mu(k)$

ie replace

$A_\mu(k)$ by $P_{\mu\nu} A_\nu(k)$

notice that $k^\mu P_{\mu\nu} A_\nu = 0$

thus gauge fields that remain

satisfy the condition $k^\mu A_\nu = 0$

ie $\partial_\mu A_\mu = 0$ again!

For these fields we can add a term

to the action of form

$$\frac{1}{2\alpha}(\partial \cdot A)^2 \sim \frac{1}{\alpha} \tilde{A}_\mu(k) k^\mu k^\nu \tilde{A}_\nu(-k)$$

path integral will be indep of α!

Furthermore can choose α so that

quadratic term is

$$\tilde{A}_\mu(k)\, g^{\mu\nu} k^2\, \tilde{A}_\nu(-k)$$

Feynman "gauge"

propagator $\boxed{D_{\mu\nu}(k) \sim \frac{g_{\mu\nu}}{k^2 - i\epsilon}}$

Seen that requirement that free spin 1 theory (massless) propagate correct # dof $\rightarrow$ Lagrangian which possesses a gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

If I then couple this theory to matter it will be crucial (unitarity again) for the resulting theory to maintain gauge symmetry

This will require that the notion of gauge transformation is enlarged so that it acts also on matter fields

Notice the following: $F_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$ suggests that introduce covariant deriv

$$D_\mu = \partial_\mu + A_\mu.$$

~~"guess" that~~

Guess that correct prescription for coupling to matter involves

$$\partial_\mu \to D_\mu .$$

But what about the transformation of matter fields?

Think about complex scalar field

$$\mathcal{L} = \int (\partial\phi)(\partial\bar{\phi}) + \dots$$

Invariant under phase transformation

$$\phi \to e^{-i\alpha}\phi \qquad \bar{\phi} \to e^{+i\alpha}\bar{\phi}$$

global symmetry

Noether current ← associated with

$$\frac{\delta \mathcal{L}}{\partial_\mu \phi}\,\delta\phi \sim -(\partial\bar{\phi})\, i\alpha\phi$$

expect must couple A_μ to conserved current

So, try

$$D_\mu\phi \to (\partial_\mu + iA_\mu + i\partial_\mu\alpha)\, e^{-i\alpha}\phi$$

$$= e^{-i\alpha}(\partial_\mu + iA_\mu)\phi = e^{-i\alpha} D_\mu\phi$$

$(\overline{D_\mu\phi})(D_\mu\phi)$ invariant now! ← MORE LATER!

Spin 1/2

Seen that constructing Lagrangians for spin $J > 0$ representations of Poincaré group is non-trivial

Reps are labelled by mass $m > 0$ & spin J ← tells us how state behaves under "little group" ← remaining symmetry when you fix momentum & jump to stationary frame ~ SO(3)

No finite dimensional unitary reps → fields

In general Lorentz covariant tensor fields describe reducible reps → need to impose additional constraints to retain unitarity.

Turns out that there is an exception to this situation for the simplest type of particle

Spin 1/2 Weyl fields ....

4~~B4~~5

Lorentz group

SO(3,1) has 6 generators

$$\Lambda \sim e^{i(\theta_i J_i + \beta_i k_i)}$$

$i = 1..3$

J's regular rotations

k's ~~boosts~~ Lorentz boosts

↑ antihermitian (non-compact)

(this is ultimately why need ∞ dim reps)

take combinations

$$J_i^+ = \frac{1}{2}(J_i + i k_i), \quad J_i^- = \frac{1}{2}(J_i - i k_i)$$

can show

$$[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-$$

$$[J_i^+, J_j^-] = 0$$

2 commuting SU(2) algebras!

$$SO(3,1) = SU(2) \oplus SU(2)$$

Irreps of $SO(3,1)$ thus labeled by (4.6)!

2 integers (A, B)

↑ has $(2A+1)(2B+1)$ d.o.f.

Notice in general that rep of Lorentz yields

several reps of $SO(3)$ (little group)

$A+B \ldots |A-B|$ (addition of ang momentum)

spin 1 + spin 0

$\equiv (\tfrac{1}{2}, \tfrac{1}{2})$ ← as we have seen!

but simplest reps of Lorentz are $(\tfrac{1}{2}, 0)$ & $(0, \tfrac{1}{2})$

yield single reps of $SO(3)$ $\sim SU(2)$

↕

Weyl field,

can be represented by Pauli matrices → spinors

2×2 matrices satisfying $SU(2)$ alg.

e $(\frac{1}{2}, 0) \equiv \left.\begin{array}{l}\vec{J}^- = \frac{1}{2}\vec{\sigma} \\ J^+ = 0\end{array}\right\}$

$(0, \frac{1}{2}) \quad \left.\begin{array}{l}\vec{J}^+ = \frac{1}{2}\vec{\sigma} \\ J^- = 0\end{array}\right\}$

<u>notice</u> under rotations $(\frac{1}{2}, 0)$ & $(0, \frac{1}{2})$
transform same way $\vec{J} = \frac{1}{2}\vec{\sigma}$

but under boosts $\vec{K} = \frac{1}{2}\vec{\sigma}$ or $-\frac{1}{2}\vec{\sigma}$

act on 2 component objects — spinors

$(\frac{1}{2}, 0)$ Lefthanded Weyl spinor

$(0, \frac{1}{2})$ righthanded " "

$$\left.\begin{array}{l}\psi_R \to e^{\frac{1}{2}(i\theta_j\sigma_j + \beta_j\sigma_j)}\psi_R \\ \psi_L \to e^{\frac{1}{2}(i\theta_j\sigma_j - \beta_j\sigma_j)}\psi_L\end{array}\right\}$$

To get ∞ dim (& hence unitary reps) introduce spinor fields $\psi_L(x)$, $\psi_R(x)$