Lecture 6

Lagrangian for Weyl fields

reminder: $\delta\psi_L = \frac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L$

$\delta\psi_L^\dagger = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j$

(infinitessimal)

Looking for Lorentz invariant kinetic term.

notice that

$$\delta(\psi_L^\dagger\psi_L) = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j\psi_L + \frac{1}{2}\psi_L^\dagger(i\theta_j - \beta_j)\sigma_j\psi_L$$

$$= -\beta_j\psi_L^\dagger\sigma_j\psi_L$$

also

$$\delta(\psi_L^\dagger\sigma_i\psi_L) = \frac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j\sigma_i\psi_L + \frac{1}{2}\psi_L^\dagger\sigma_i(i\theta_j - \beta_j)\sigma_j\psi_L$$

$$= \frac{1}{2}-i\theta_j\psi_L^\dagger(\sigma_j\sigma_i - \sigma_i\sigma_j)\psi_L$$

$$-\frac{\beta_j}{2}(\sigma_j\sigma_i + \sigma_i\sigma_j)\psi_L^\dagger\psi_L$$

$$= -\frac{i}{2}\theta_j \psi_L^\dagger 2i\epsilon_{jik}\sigma_k \psi_L$$

$$- \beta_i \psi_L^\dagger \psi_L$$

ie

$$\delta(\psi_L^\dagger \sigma_i \psi_L) = \epsilon_{jik}\,\theta_j\, \psi_L^\dagger \sigma_k \psi_L - \beta_i \psi_L^\dagger \psi_L$$

thus $(\psi_L^\dagger \psi_L\,,\ \psi_L^\dagger \sigma_i \psi_L)$ transforms as 4-vector!

$\therefore\ \mathcal{L} = i\psi_L^\dagger \bar{\sigma}_\mu \partial^\mu \psi_L$ is Lorentz invariant

where $\bar{\sigma}_\mu = (I, \vec{\sigma}_i)$

Similarly $i\psi_R^\dagger \sigma_\mu \partial^\mu \psi_R$ is also Lorentz invariant

where $\bar{\sigma}_\mu = (I, -\vec{\sigma}_i)$

$\sim$

Can combine these 2 objects into

<u>Dirac</u> spinor

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

if write $\bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}$

& $\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma_\mu & 0 \end{pmatrix}$

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi$$

easy to show that $\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$ also L.I

$= \underline{\bar{\psi}\psi}$ ← check <u>this</u>!

yielding finally

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi \quad \text{Dirac action.}$$

ψ transforms as sum of $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ irreps of Lorentz symmetry.

Useful things about Dirac spinors

Lorentz generators $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$\bar{\psi}\partial_\mu\psi$, $\bar{\psi}\gamma_\mu\psi$ transforms like 4 vector.

$\bar{\psi}\gamma_\mu\gamma_\nu\psi$ tensor.

$\bar{\psi}\gamma_5\psi$ pseudoscalar (changes sign under parity)

$\bar{\psi}\gamma_5\gamma_\mu\psi$ pseudovector

etc. $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ $\quad \{\gamma_5, \gamma_\mu\}_+ = 0.$

$$\left.\begin{aligned} \psi_L &= \tfrac{1}{2}(1-\gamma_5)\psi \\ \psi_R &= \tfrac{1}{2}(1+\gamma_5)\psi \end{aligned}\right\}$$

<u>notice</u>: $\psi^\dagger\psi$ is <u>not</u> Lorentz invariant

instead it is the time component of a current

$\bar{\psi}\gamma_\mu\psi$ ← corresponds to Noether current associated with phase invariance

$\psi \to e^{i\alpha}\psi$ ← conservation of fermion #

Notice that the simplest fermion action is built from single Weyl spinor

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}_\mu \partial_\mu \psi_L + i\frac{M}{2}\left(\psi_L^\dagger \sigma_2 \psi_L^* - \psi_L^T \sigma_2 \psi_L\right)$$

Majorana mass term.
It is Lorentz invariant

(check...)

notice I can generate this mass term from a Dirac spinor of the form

$$\begin{pmatrix} \psi_L \\ i\sigma_2 \psi_L^* \end{pmatrix} \leftarrow \text{Majorana spinor}$$

notice that $i\sigma_2 \psi_L^*$ must transform like a R handed ~~spinor~~ Weyl spinor

again one can show that charge conjugate

$$-i\sigma_2 \psi^* = \psi = \psi_c \quad \text{for such a spinor}$$

Thus ψ cannot be charged under $U(1)$ ← real reps only...

Fermionic path integrals

To formulate P.I you need to allow for Pauli principle – ~~that~~ fermions anticommute

The P.I must be defined for integration over (classical) anticommuting numbers

+ i.e no operators

→ Grassman #s.

$$\theta_i \theta_j = -\theta_j \theta_i \qquad \leftarrow \text{so } \boxed{\theta_i^2 = 0}$$

Most general function of single Grassman

is $f = a + b\theta$

for 2

$$f = a + b\theta_1 + c\theta_2 + d\theta_1\theta_2$$

What about integrals?

want integrals to be invariant under shifts in integration variable (to complete square...)

$$\int f(\eta)\, d\eta = \int f(\eta + \xi)\, d\eta$$

$\longrightarrow \int d\eta \; b\xi = 0 \;$ i.e. $\int d\eta = 0$

Also define $\int d\eta\, \eta = 1$.

Grassmann even.

eg/ $\int d\eta_1 d\eta_2 (a + b\eta_1 + c\eta_2 + d\eta_1\eta_2)$

$= d$!

be careful with order of integrations

$\int d\eta_1 d\eta_2 = -\int d\eta_2 d\eta_1$.

For N variables:

$\int d\eta_1 \dots d\eta_N\, f(\eta_1 \dots \eta_N)$ coeff of $\eta_1 \dots \eta_N$

$= d_{1\dots N}$

totally antisymmetric matrix.

exponentials:

$\int d\eta_1 \dots d\eta_N\, e^{\eta_i M_{ij} \eta_j}$ $M = -M^T$ obviously

expand

$(1 + \eta_i M_{ij} \eta_j + \frac{1}{2} \eta_i M_{ij} \eta_j \eta_k M_{kl} \eta_l + \dots)$

only non-zero terms when N η's appear.

i.e. $(\eta M \eta)^{N/2} / (\frac{N}{2})!$ piece.

$$\int d\eta_1 \cdots d\eta_N \, \eta_{i_1} \cdots \eta_{i_N} \, M_{i_1 i_2} \cdots M_{i_{N-1} i_N}$$

antisymmetric $\qquad \times \left(\frac{1}{N/2}\right)!$

$\propto \epsilon_{i_1 \cdots i_N}$ after integration

$$\Rightarrow \int D\eta \, e^{\eta M \eta} = \epsilon_{i_1 \cdots i_N} M_{i_1 i_2} \cdots M_{i_{N-1} i_N}$$

$$= Pf(M)$$

$\boxed{\text{Pfaffian}}$

Complex Grassmann

$$\theta = \xi + i\eta, \quad \bar{\theta} = \xi - i\eta$$

$$\int D\theta D\bar{\theta} \, e^{\bar{\theta} M \theta}$$

$$= \int d\xi \, d\eta \, e^{\xi M \xi + \eta M \eta}$$

$$= [Pf(M)]^2 !$$

$Pf^2 = \det$

expanding integr directly

$$\int D\theta D\bar{\theta} \left(\cdots \frac{(\bar{\theta} M \theta)^N}{N!} \cdots \right) = \frac{1}{N!} \epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} M_{i_1 j_1} \cdots M_{i_N j_N}$$

$$= \det(M)$$

Some more bits of technology

— need to be able to differentiate to include (fermionic) sources

eg if $f(\psi) = a\psi$

$$\frac{\delta}{\delta\psi} f = a \quad \leftarrow \text{same as integration!}$$

That's it! now back to writing down the partition function for theory of fermions.

Dirac action

$$S = \int d^4x \, \bar{\psi} (i\not{\partial} - m) \psi$$

$$Z(\eta, \bar{\eta}) = \int D\psi D\bar{\psi} \, e^{iS + i\bar{\eta}\psi + i\bar{\psi}\eta}$$

again complete square $\Rightarrow$

$$\bar{\theta} = \bar{\psi} + \bar{\eta} M^{-1}$$

$$\theta = \psi + M^{-1}\eta$$

$$\rightarrow S = i\bar{\theta} M \theta - i\bar{\eta} M^{-1} \eta.$$

$$\therefore\ Z(\eta,\bar\eta) = \det M\, e^{i\bar\eta_i M^{-1}_{ij}\eta_j}$$

lattice

$$\therefore\ G^{2c}_{ij} = \frac{\delta^2}{\delta\bar\eta_i\,\delta\eta_j}\ln Z(\eta,\bar\eta)$$

$$= M^{-1}_{ij}$$ ← fermion propagator.

In continuum

$$M_{ij} \to i\not\partial - m$$

$$\delta_{ij} \to S(x-y) \text{ satisfying}$$

$$(i\not\partial - m)\,S(x-y) = \delta^4(x-y)$$

In k space:

$$(\not p - m)\,S(p) = 1$$ ← unit matrix in spinor space.

$$\therefore\ S(x) = \int \frac{d^4p}{(2\pi)^4}\,\frac{(\not p + m)}{p^2 - m^2 + i\epsilon}$$

↑ ensures causal propagation as before.