Homework 3

① Show that $\dfrac{1}{AB} = \displaystyle\int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$

As discussed in class the polarization tensor is given by the following expression at 1 loop:

$$i\Pi_{\mu\nu}(k) = -e^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left(S(l+k)\gamma_\mu S(l)\gamma_\nu \right)$$

Use the above result to rewrite this as

$$i\Pi_{\mu\nu}(k) - e^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{4N_{\mu\nu}}{(q^2+D)^2}$$

where

$$D = x(1-x)k^2 + m^2 \quad (\text{Eucl. space})$$

& $q = l + xk$.

& $4N_{\mu\nu} = \mathrm{Tr}\left[(-\not{l} - \not{k} + m)\gamma_\mu (-\not{l} + m)\gamma_\nu\right]$

Simplify the trace terms using properties of the γ matrices

Now generalize integral to d dimensions & prove the following results ← putting $e = \mu^{\epsilon/2} e$

$$\int d^d q \, q^\mu f(q^2) = 0$$

$$\int d^d q \, q^\mu q^\nu f(q^2) = C_2 g^{\mu\nu} \int d^d q \, q^2 f(q)$$

where C_2 is to be determined.

Finally, using the result

$$\int \frac{d^d q}{(2\pi)^d} \frac{(q^2)^a}{(q^2+D)^2} = \frac{\Gamma(2-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(2)\Gamma(\tfrac{1}{2}d)} \times D^{-(+2-a-d/2)}$$

where Γ is Euler gamma function

$$\Gamma(n+1) = n!$$

$$\Gamma(n+1/2) = \frac{(2n)!}{n!\,2^{(2n)}}\sqrt{\pi}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left(\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + O(x)\right)$$

show that

$$i\Pi_{\mu\nu} = -k^2 P_{\mu\nu} \frac{e^2}{\pi^2} \int dx \, x(1-x)\left[\frac{1}{\epsilon} - \frac{1}{2}\ln D/\mu^2\right]$$

② The exact fermion (inverse) propagator in QED takes the form

$$S(p)^{-1} = \not{p} + m - \Sigma(\not{p})$$

In an on-shell renormalization scheme m should be the physical (pole) mass & the residue at the pole should be 1 (ie the propagator should behave like $\frac{1}{\not{p} + m}$)

This implies the normalization conditions $\Sigma(-m) = 0$ & $\Sigma'(-m) = 0$

At 1 loop $\Sigma(p)$ is given by

$$i\Sigma(\not{p}) = -\frac{e^2}{8\pi^2}\int_0^1 dx\left[(1-x)(2-\epsilon)\not{p} + (4-\epsilon)m\right] \times \left(\frac{1}{\epsilon} - \frac{1}{2}\ln D/\mu^2\right)$$

where $D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$

& μ is a scale introduced via dimensional regularization to keep e^2 dimensionless

$$e^2 \to e^2 \mu^{\epsilon/2} \quad \& \quad \epsilon = 4-d$$

The first condition $\Sigma(-m) = 0$ can be ensured by writing

$$\Sigma(\not{p}) = \frac{e^2}{8\pi^2}\left[\int dx\, \left((1-x)\not{p} + 2m\right) \ln D/D_0 + k_2(\not{p}+m)\right]$$

where $D_0 = x^2m^2 + (1-x)\,m_\gamma^2$ ie D at $\underline{p^2 = -m^2}$

Show that $\Sigma'(-m) = 0$ determines k_2

(3) Write down the Feynman diagram giving the 1 loop vacuum polarization tensor in <u>three</u> spacetime dimensions

Imagine expanding $\Pi_{\mu\nu}(k)$ about $k=0$ & examine the first term in this Taylor expansion:

$$\Pi_{\mu\nu}(k) = F_{\mu\nu\lambda}\, k_\lambda$$

Show that $F_{\mu\nu\lambda} \propto e^2 \epsilon_{\mu\nu\lambda}$ independent of the fermion mass!

What is the structure of this term in the quantum action?

ie what effective photon propagator is induced through these virtual fermion loops in 3D?