Lecture 10

Last time wrote $\mathcal{L}$:

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F_{\mu\nu}^2 - i Z_2 \bar{\psi} \not{\partial} \psi - Z_m m \bar{\psi}\psi$$
$$- i Z_1 \bar{\psi} \not{A} \psi$$

where $Z_i = 1 + O(e^2)$ & chosen to cancel off divergences arising from loops

Furthermore physical observables computed using this $\mathcal{L}$ are indep of μ.

(artificial scale arising in dim reg)

Trivially rewrite as

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}^{0\,2} - i \bar{\psi}_0 \not{\partial} \psi_0 - m_0 \bar{\psi}_0 \psi_0$$
$$- i \bar{\psi}_0 \not{A}_0 \psi_0$$

bare parameters / fields

$\underline{m_0, \psi_0, \text{etc indep of } \mu}$, likely ∞ but yield <u>finite</u> observables!

Formally

$$A_0 = \sqrt{Z_3}\, A$$
$$\psi_0 = \sqrt{Z_2}\, \psi$$
$$m_0 = Z_m / Z_2 \; m$$
$$e_0 = \left(\frac{Z_1}{Z_2\sqrt{Z_3}}\right) e \mu^{\epsilon/2}, \quad \epsilon = 4 - d.$$

notice: A, ψ, m finite renormalized quantities (cut-off indep) but in general they do depend on μ.
but m_0, A_0, ψ_0 are μ indep (but likely ∞!)

$$Z_1 = 1 - e^2/8\pi^2 \, \frac{1}{\epsilon}$$
$$Z_2 = 1 - e^2/8\pi^2 \, \frac{1}{\epsilon}$$
$$Z_3 = 1 - e^2/6\pi^2 \, \frac{1}{\epsilon}$$
$$Z_m = 1 - e^2/2\pi^2 \, \frac{1}{\epsilon}$$

$Z_1 = Z_2$ again

ignoring finite parts

eg $\overline{MS}$ scheme

Taking logs: $\Rightarrow$

$$\ln e_0 = \ln\left(\frac{Z_1}{Z_2\sqrt{Z_3}}\right) + \ln e + \frac{\epsilon}{2}\ln\mu$$

$$= \frac{e^2}{12\pi^2}\,\frac{1}{\epsilon} + \ln e + \epsilon/2 \ln\mu.$$

$$\frac{\partial \ln e_0}{\partial \ln\mu} = 0 \Rightarrow$$

$$0 = \left(1 + \frac{1}{\epsilon}\frac{e^2}{6\pi^2}\right)\frac{\partial e}{\partial \ln\mu} + \frac{\epsilon e}{2}$$

writing $$\frac{\partial e}{\partial \ln\mu} = -\frac{\epsilon e}{2} + \beta(e)$$

$\uparrow$ Beta function

$$\left(-\epsilon e/2 + \beta\right)\left(1 + \frac{1}{\epsilon}\frac{e^2}{6\pi^2} + \cdots\right) = 0$$

terms of order ϵ^0 $\rightarrow$

$$\beta(e) - e^3/12\pi^2 = 0$$

Tells you how renormalized charge must vary with parameter μ to keep physical observables μ independent.

Aside: what about the finite parts in Z_i's?

eg β function repeat with

eg $Z_i = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + f(\mu)$

$$\left(1 + \frac{1}{\epsilon}\frac{e^2}{6\pi^2} + f(\mu)\right)\left(-\epsilon e/2 + \beta\right)$$

$f(\mu)$ does not contribute at $O(\epsilon^0)$

$\hookrightarrow$ β, γ independent of finite parts in $\boxed{d=4}$ $(\epsilon = 0)$

Explains why $\overline{MS}$ scheme so attractive. Easiest to just get divergent pieces from Feynman integrals. All one needs for β, γ etc.

(disadvantage: pole of propagator no longer at physical fermion mass ...)

β function $\geqslant 0$

charge increases with energy scale μ.

(Landau pole again)

Can apply same tricks to other renormalized / bare parameters eg mass

$$\ln m_0 = \ln(Z_m/Z_2) + \ln m$$

$$= -e^2/2\pi^2 \frac{1}{\epsilon} + \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \ln m$$

$$\frac{\partial \ln m_0}{\partial \ln\mu} = 0 \Rightarrow \quad -\frac{3}{8\pi^2}\frac{1}{\epsilon}\frac{2e\,\partial e}{\partial \ln\mu}$$

$$+ \frac{1}{m}\frac{\partial m}{\partial \ln\mu} = 0$$

now $\dfrac{\partial e}{\partial \ln\mu} = -\epsilon e/2 + \beta$

define

mass anomalous dimension $\gamma_m = \dfrac{1}{m}\dfrac{\partial m}{\partial \ln\mu}$

$$\therefore \quad \gamma_m = \frac{3}{4\pi^2}\frac{1}{\epsilon} e\left(\beta - \epsilon e/2\right)$$

at $O(\epsilon^0)$: $\Rightarrow$

$$\gamma_m = \frac{-3}{8\pi^2} e^2 + O(e^4)$$

Similarly can define anomalous dimensions of fields ψ, A

$$\frac{1}{2}\partial/\partial \ln\mu \; Z_2 = \gamma_\psi(e)$$

$$\frac{1}{2}\partial/\partial \ln\mu \; Z_3 = \gamma_A(e)$$

Furthermore <u>bare Green functions</u>

$$\Gamma_0^{(n)} = \langle \psi_0 \ldots \bar{\psi}_0 \ldots A_0 \rangle$$

are also <u>indep of μ</u>

use this fact to derive <u>differential eq^ns</u>

<u>renormalized</u> Green functions

<u>Callan-Symanzik</u> eq^ns (RG eq^ns)

eg $\langle \bar{\psi}_0 \psi_0 \rangle = \Delta_0(k^2)$

$$\frac{\partial}{\partial \ln\mu} \ln \Delta_0(k^2) = 0$$

but $\Delta_0(k^2) = Z_2 \Delta(k^2)$

$$\therefore\ 0 = \frac{\partial \ln Z_2}{\partial \ln\mu} + \frac{1}{\Delta}\left(\frac{\partial}{\partial \ln\mu} + \frac{\partial e}{\partial \ln\mu}\frac{\partial}{\partial e} + \frac{\partial m}{\partial \ln\mu}\frac{\partial}{\partial m}\right)\Delta(k^2)$$

since Δ depends on e, m & μ.

or $$\left(\frac{\partial}{\partial \ln\mu} + \beta(e)\frac{\partial}{\partial e} + \gamma_m m\frac{\partial}{\partial m} + 2\gamma_\psi\right)\Delta(k^2) = 0$$

<u>notice</u>: if $m=0$ & $\beta=0$ (CFT)

$$\left(\frac{\partial}{\partial \ln\mu} + 2\gamma_\psi\right)\Delta = 0$$

~~ie~~

$$\therefore\ \Delta \sim \mu^{-2\gamma_\psi} = (\mu^2)^{-\gamma}$$

or putting in dimensions

$$\Delta \sim (\mu/k)^{-2\gamma_\psi}$$

but we know

$\Delta(k^2) \sim \frac{1}{k^2}$

(invisible in diff eq wrt μ)

$\Delta \sim \frac{1}{k^2}\left(\frac{\mu^2}{k^2}\right)^{-\gamma}$

ie

$2 \to 2 - 2\gamma$

under k scaling

Why the term anomalous dimensions?

consider scalar field theory

$$S = \int d^4x \left(-\varphi \Box \varphi + m^2 \varphi^2 + \lambda \varphi^4 \right)$$

Invariant under dilation / scale symmetry:

$$x \to 1/\lambda \quad \partial \to \lambda \partial \quad m \to \lambda m \quad g \to g \quad \varphi \to \lambda \varphi$$

Same argument earlier.

$$G^n = \langle \varphi_1 \cdots \varphi_n \rangle \to \lambda^n G^n \quad \underline{\text{classically}}$$

however OM the scale μ appears in renormalized G's.

$$G^n(x, g, m, \mu) \sim m^a g^b x_1^{c_1} \cdots x_n^{c_n} \mu^\gamma$$

classically $a - c_1 - \cdots c_n = n$ classical scaling dimension of G^n.

qm: $a + \gamma - c_1 - \cdots c_n = n$

or $a - c_1 \cdots - c_n = \underline{n - \gamma}$

thus under dilatations $G^n \to \lambda^{n-\gamma} G^n$

since μ does not ~~scale~~ transform under dilations

thus classical scaling dim shifts $n \to n - \gamma$

$$\gamma \cdot \partial G^n / \partial \ln \mu = \gamma \quad \text{as above}$$

Further comments on renormalization

RG

* QED is an example of a <u>renormalizable</u> QFT
* This is just the statement that a finite # of counterterms are needed to remove all ∞'s to all orders of p. theory
* Standard arguments tell you that this property is tied to the fact ~~that~~ the $\mathcal{L}$ contains no interactions with <u>negative</u> mass dimension couplings
* Actually ~~=~~ 4d QED (& its non-abelian cousins) pretty much saturate the class of renormalizable QFT (at least perturbatively)

* Key idea: quantum effects can strongly affect values of parameters in classical Lagrangian

Indeed the classical parameters are in principle unobservable & the best one can hope to do is to reparametrize theory in terms of renormalized parameters whose values are ultimately taken from expt.

* Problems all stem from short distance scales (UV) & an equivalent way to think of the procedure of renormalization is ~~that~~ it is a systematic way of rewriting parameters of theory in ~~way~~ such a way as to render low energy observables insensitive to UV scales

* i.e physics at one energy scale should not be sensitive to physics at some much higher scale

Price you pay

* Removing this sensitivity to U.V scales / cut off, leaves a remaining imprint: the coupling constants are not constant!

They change with physical scale (or μ)

$\ln \mu^2 \rightarrow \ln \mu^2/k^2$

* this is seen most graphically in the running of the interaction strength with momentum scale $e(\mu)$ or $e(k)$

— β function etc

anomalous scaling of correlation functions via γ.

Important part

* Many of these comments also apply to so-called non-renormalizable theories

* In principle such theories require a ∞ # counterterms if one wants to work to all orders in the coupling.
  However to low order the non-renormalizable interactions only contribute terms $O\left(E/\Lambda\right)^n$ when Λ is some scale needed to write down the non-renormalizable operators in the classical theory

* Perturbation theory breaks down when $E \to \Lambda$ but if $E \ll \Lambda$ the theory can yield precise predictions for physical observables including loop effects.

  "effective field theories"

Wilson & condensed matter

* Indeed this is precisely what happens in CMP. In that case one is interested in phenomena at distance scales much greater than the lattice spacing $\left(\frac{1}{\Lambda}\right)$

* In this case details of $\mathcal{L}$ at short distances including all sorts of non-renormalizable ops are irrelevant to the behavior of system at large scales (hence the term irrelevant op.)

* Related to concept of Universality: different microscopic $\mathcal{L}$ all lead to same long distance physics (controlled by the renormalizable or relevant ops)

* Wilson made all this clear and developed an alternative to the continuum RG which is in many ways more intuitive & plausible ...

Wilson's key idea

① Think of your theory on a lattice with cut-off $\Lambda = 1/a$

② Long distance physics should be insensitive to Λ

③ You should be able to change the "bare" parameters of theory to compensate for changes in Λ → leave long distance physics invariant

④ In practice this long distance universality only occurs if one is on a critical surface in space of bare parameters. Corresponds to tuning all relevant parameters eg masses to ~~the~~ special values (eg zero!)

⑤ On such a critical surface there may occur fixed pts ~~where~~ where physics is scale invariant ie $\beta = 0$. CFTs

* Critical exponents in CMP are related to γ's in QFT & characterize the RG flows close to one of these fixed pts

* Wilson & others developed series of technologies

1. ε Epsilon-expansion: keep $\epsilon = 4-d$ finite & expand in $\epsilon \to d=3$) eg Wilson-Fisher
2. exact RG equations express Λ indep of physical observables (later?)
3. Numerical / Monte Carlo implementations of RG in real space

Powerful techniques which have taught us a lot about strongly interacting QFTs