Lecture 11

Non-abelian gauge theory

Saw that in QED invariance under

$$\psi \to e^{-ie\alpha(x)}\psi$$

provided all derivatives replaced by covariant derivatives

$$D_\mu = \partial_\mu - ieA_\mu .$$

need ensures $D_\mu\psi \to e^{ie\alpha} D_\mu\psi$

ie $$D_\mu \to e^{-ie\alpha(x)} D_\mu e^{ie\alpha(x)}$$

thus $$D_\mu' = e^{-ie\alpha(x)}\partial_\mu e^{ie\alpha(x)}$$

$$+ie\, e^{-ie\alpha} A_\mu e^{ie\alpha}$$

in QED $\Longrightarrow$ $$D_\mu' = \partial_\mu + ie\partial_\mu\alpha + ieA_\mu$$

$$= \partial_\mu - ieA_\mu'$$

ie $$A_\mu' = A_\mu - \partial_\mu\alpha$$

Can easily generalize to higher symmetries

Eg imagine ψ take values in some irrep of

$SU(N)$

Action of group on ψ implemented by

matrices $U(x)$ (unitary)

eg $\quad e^{-i\alpha} \longrightarrow U(x) \sim e^{-ig\theta}$

$\quad U(1) \qquad\qquad SU(N) \qquad \theta^+ = \theta.$

Theory invariant under these local rotations

internal space iff

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$$

(8) $\quad D_\mu \to U(x) D_\mu U^+(x) \qquad *$

$$\text{if } \psi \to U(x)\psi$$

requires $U^+U = I$ unitary matrix.

Often interested in case $SU(N)$

where $\det U(x) = 1$

matrix valued

$$* \Rightarrow \quad A_\mu \to U(x) A_\mu U^+(x) + \frac{i}{g} U(x) \partial_\mu U^+(x)$$

In general unitary matrix can be written

$$\mathcal{U}(x) = e^{-ig \sum_{a=1}^{N_G} \theta^a(x) T^a}$$

where N_G = # generators of group

eg $N_G = N^2 - 1$ for $SU(N)$

T^a must be hermitian & traceless in this case.

The generators must satisfy Lie algebra

$$[T^a, T^b] = i f^{abc} T^c$$

structure constants

for infinitesimal transformations

$$\mathcal{U} \simeq 1 - ig\,\theta^a T^a = I - ig\,\theta \cdot T$$

$$A_\mu \to (1 - ig\theta \cdot T) A_\mu (1 + ig\theta \cdot T)$$
$$+ \, {}^i/_g (1 - ig\theta \cdot T) \partial_\mu (1 + ig\theta \cdot T)$$

ie $A_\mu \to A_\mu - ig[\theta, A_\mu] - \partial_\mu \theta$; $\delta A_\mu = -D_\mu \theta$

defined covariant derivative acting on

adjoint representation

$$\delta A_\mu = -(\partial_\mu \theta + g f^{abc} \underbrace{\theta^b A_\mu^c}_{})$$

$$\delta A_\mu^a = \left(e^{(f_{abc}\theta_b)} - I\right) A_\mu^c$$

ie A_μ^a transform under group (in adj rep) in that rep where generators are structure constants themselves. Notice this is a real rep. since f's are real (δA_μ^a)

In QED can define $F_{\mu\nu}$ as

$$\frac{i}{e}[D_\mu, D_\nu]$$

Similarly in non-abelian case define

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

matrix valued

new term

Notice now $F_{\mu\nu} \to U F U^\dagger$ under G.T.

$\therefore$ kinetic term must be

$$-\frac{1}{2} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \leftarrow \underline{\text{Gauge invariant}}$$

writing $A_\mu = A_\mu^a T^a$

$$\therefore \mathrm{Tr}(A_\mu T^b) = A_\mu^a \underbrace{\mathrm{Tr}(T^a T^b)}_{\substack{\text{conventionally} \\ \frac{1}{2}\delta^{ab}}}$$

Similarly $F_{\mu\nu}^a = 2\mathrm{Tr}(F_{\mu\nu} T^a)$

kinetic term may also be written

$$-\frac{1}{4} \mathrm{Tr}\left(F_{\mu\nu}^a F^{\mu\nu a}\right) \quad \text{c.f } \underline{\text{QED}}$$

where

$$F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b$$

Points to note

* $\mathcal{L}$ now contains 3, 4 pt self-interactions of gauge field — pure non-abelian theory is <u>non-trivial</u>.

* Play same game for other groups eg $SO(N)$, $Sp(2N)$ - -

* Important ~~that~~ these groups are <u>compact</u> $Tr(T^A T^B)$ pos def matrix (metric on group space)

$$Tr(T^A T^B) = \frac{1}{2}\delta^{AB} \quad \text{fundamental rep.}$$

* QCD, EW theory based on $SU(3)$ & $SU(2) \times U(1)$ groups

New feature arises in YM theory
not present in QED

Can find other sets of matrices satisfying

$$[T^a_R, T^b_R] = i f^{abc} T^c_R$$

not just $N \times N$ defining / fundamental

These other generators mean that matter fields can transform in other representations of the group (algebra)

Only thing we have to do is to modify D_μ

$$D_\mu = \partial_\mu - i g T^a_R A^a_\mu(x)$$

acts on R component object

dim of rep = size of matrices T_R.

eg/ adjoint rep

$$T^a_A = -i f^a{}_{bc}$$

$$\dim = N^2 - 1 = \# \text{ generators}$$

Also, from

$$[T^a, T^b] = i f^{abc} T^c$$

take complex conjug

↳ $-T^{a*}$ satisfies same Lie algebra

In general $-T^{a*}$ diff rep from fundamental

— called (anti) fundamental or

complex conjugate rep.

Sometimes $-T^{a*} = T^a$ — rep is called real

(adjoint is one real example)

In ~~these~~ cases T's are all pure imaginary entries....

also possible that

$$-T^{a*} \sim U T^a U^\dagger \qquad U\text{-unitary matrix}$$

pseudoreal e.g. SU(2) has this property.

Rep^{n}s can be classified by certain #'s

1. Quadratic Casimir $\sum T^a T^a = C_2(R)$
(commutes with all generators c.f L^2 with L_x, L_y, L_z and $SU(2)$)

2. trace invariant (index)

$$\mathrm{Tr}(T^a T^b) = T(R)\,\delta^{ab}$$

$= 1/2$ fundamental.

$= N$ for adjoints

note $C_2(R) = N$ adjoints and $\frac{(N^2-1)}{2N}$ fund.

In general can build new representations by taking direct product — reducible

eg $SU(N)$ $\quad N \otimes \bar{N} = 1 \oplus A$

Irreducible rep^{n}s

Other useful invariants

classifying different reps

$$T(R)\, f^{abc} = -i\,\mathrm{Tr}\left(T^a_R\,[T^b_R, T^c_R]\right)$$

⊕ completely sym tensor.

$$A(R)\, d^{abc} = \frac{1}{2}\mathrm{Tr}\left(T^a_R\,\{T^b_R, T^c_R\}\right)$$

anomaly coefficient of rep

$$A(\bar{R}) = -A(R)$$

if R is real or pseudoreal

$$\underline{A(R) = 0}$$

for $SU(N)$ $N > 3$

$$A(\text{fund}) = 1.$$