<u>Lecture 12</u>

Path integral for non-abelian gauge theory. (9)

Initially no fermions

$$Z(J) = \int DA\, e^{i S_{YM}(A,J)}$$

$$S_{YM}(A,J) = \int d^4x \left(-\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + J^{\mu a}_{\mu} A^a_\mu \right)$$

In QED noted that can define a propagator only after having removed configurations from path integral which in k space have component $\parallel k_\mu$.

Relied on form of gauge transformation

$$A_\mu \to A_\mu - \underbrace{\partial_\mu}_{k_\mu \text{ in } k\text{-space}} \alpha$$

but in non-abelian theory transformation is non-linear — will not do!

in fact

$$\delta A_\mu = - D_\mu \alpha$$

non-abelian case

transformation depends on A_μ itself

Toy model

$$Z = \int dx\,dy\, e^{iS(x)}$$

since integrand does not depend on y can drop the $\int dy$. This is basically what we did for QED to remove gauge redundancy.

Equivalently

$$Z = \int dx\,dy\,\delta(y)\,e^{iS(x)}$$

or

$$= \int dx\,dy\,\delta(y-f(x))\,e^{iS(x)}$$

← analog of gauge condition

rewrite implicitly

$$y - f(x) = 0 \equiv G(x,y) = 0$$

$$\delta(G(x,y)) = \frac{\delta(y-f(x))}{|\partial G/\partial y|}$$

$\hookrightarrow$

$$Z = \int dx\,dy\,\frac{\partial G}{\partial y}\,\delta(G)\,e^{iS}$$

N variables

$$Z = \int d^N x\, d^N y\, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i)\, e^{iS(x)}$$

To treat gauge theories let

$$\left.\begin{array}{l} x \equiv A_\mu(x) \\ y \equiv \theta(x) \end{array}\right\}$$

$G \equiv$ gauge fixing function

consider

$$G^a(x) = \partial^\mu A^a_\mu - \omega^a(x)$$

$\uparrow$ arbitrary

(R_ξ gauge)

$$\Longrightarrow \quad Z(J) \propto \int DA \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G) e^{iS_{YM}}$$

under infinitesimal G.T.

$$G^a(x) \to G^{(a)}(x) - \partial_\mu D_\mu^{ab} \theta^b$$

$$\therefore \quad \frac{\delta G^a(x)}{\delta \theta^b(y)} = -\partial^\mu D_\mu^{ab} \delta^4(x-y)$$

still tricky but remember that det's result from integration over grassman fields

here $\quad c^a(x)\,,\quad \bar{c}^a(x) \quad$ Fadeev-Popov ghosts

$$\det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int Dc D\bar{c}\, e^{iS_{gh}}$$

$$S_{gh} = \int d^4x \mathcal{L}_{gh}\ \&$$

$$\mathcal{L}_{gh} = \bar{c}^a \partial^\mu D_\mu^{ab} c^b$$

$$= -\partial^\mu \bar{c}^a D_\mu^{ab} c^b \quad \text{int by parts}$$

$$= -\partial^\mu \bar{c}^a \partial_\mu c^a + ig\, \partial^\mu \bar{c}^a A_\mu^c T_A^{cab} c^b$$

$$= -\partial^\mu \bar{c}^a \partial_\mu c^a + g f^{abc} A_\mu^c \partial^\mu \bar{c}^a c^b$$

free action for complex, anticommuting scalar field

(derív) int- with gauge field

in abelian theory $f^{abc} = 0$

$\longrightarrow$ free field term only $\rightarrow$ drop (as we did!)

Since c violates spin-statistics thm cannot appear as external ptcles ... later ...

One final trick ...

$G^a(x)$ contains $\omega^a(x)$

but <u>Z indep of ω.</u> Therefore can multiply $Z(J)$ by arbitrary function of ω & integrate over ω

eg/ $$e^{-i/2\xi \int d^4x\, \omega^a(x)^2}$$

$\hookrightarrow$

$$Z(J) = \int DA\, Dc\, D\bar{c}\; e^{i(S_{YM} + S_{Gh} + S_{gf})}$$

where $S_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$.

$\uparrow$ Lorentz gauge fixing term!

needed to allow us to <u>do perturbation theory</u> by defining invertible quadratic action for A_μ ...

Non-abelian gauge theory

Feynman rules:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu}{}^{a} F^{\mu\nu a}$$

$$= -\tfrac{1}{4}\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g f^{abc} A^{\mu b} A^{\nu c}\right) \times$$

$$\left(\partial_\mu A_\nu{}^{a} - \partial_\nu A_\mu{}^{a} + g f^{ade} A_\mu{}^{d} A_\nu{}^{e}\right)$$

$$= -\tfrac{1}{\cancel{4}2}\, \partial^\mu A^{\nu a} \partial_\mu A_\nu{}^{a} + \tfrac{1}{2}\, \partial^\mu A^{\nu a} \partial_\nu A_\mu{}^{a}$$

$$- g f^{abc} A^{\mu b} A^{\nu c} \partial_\mu A_\nu{}^{a} \leftarrow (A)$$

$$- \tfrac{1}{4} g^2 f^{abc} f^{ade} A^{b\mu} A^{c\nu} A_\mu{}^{d} A_\nu{}^{e}$$

also gauge fixing term:-

$$-\frac{1}{2\xi}\, \partial_\mu A^{\mu a} \partial^\nu A_\nu{}^{a}$$

ie $\mathcal{L}_{YM} + \mathcal{L}_{GF} = \frac{1}{2} A^{a\mu}\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right) A^{a\nu}$

$$+ \frac{1}{2\xi} A^{a\mu} \partial_\mu \partial_\nu A^{a\nu} + \text{int}^{\text{ns}}$$

Thus "gluon" propagator

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon} \times \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{\xi k_\mu k_\nu}{\xi k^2} \right)$$

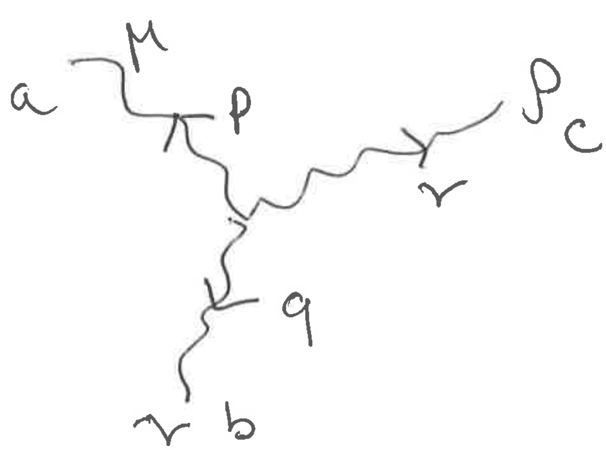
if $\xi = 1$ (Feynman gauge)

$$\Delta^{ab}_{\mu\nu}(k) = \frac{\delta^{ab} g_{\mu\nu}}{k^2}$$

Interactions

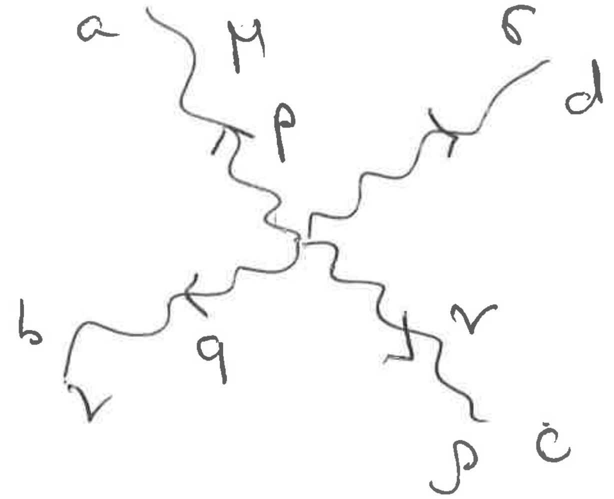
3 gluon vertex:

$$i(-g f^{abc})\left[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu} \right]$$



↑ to see this integrate expression by parts to put deriv on different A's in ✱

4 gluon vertex?



$$-ig f^{abe} f^{cde} g_{\mu\rho} g_{\nu\sigma} + \text{perms.}$$

For loop calcs also need ghost terms

$$\mathcal{L}_{gh} = -\partial^\mu \bar{c}^b D_\mu^{bc} c^c$$

$$= -\partial^\mu \bar{c}^c \partial_\mu c^c + g f^{abc} A_\mu^a \partial^\mu \bar{c}^b c^c$$

ghost propagator

$$\Delta^{ab}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}$$

ghost-gauge vertex

$$ig f^{abc}(-iq_\mu) = g f^{abc} q_\mu .$$

Quarks / fermions

gauge or "color" indices

$$\mathcal{L}_q = i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi$$

quark propagator $\dfrac{(-\not{p}+m)\,\delta_{ij}}{p^2+m^2-i\epsilon}$

quark-gluon vertex

$ig\gamma^\mu (T^a_R)_{ij}$ $\quad$ i, j run from $1 \ldots d(R)$

dimension of rep of group.

Tree level calcs are already hard just because of the proliferation of indices...

eg $gg \to gg$ cross-section needs 12,996 terms!

Special tricks exist for handling this & reducing # terms – but we won't discuss it further.

Instead consider one loop structure of these theories ...