Lecture 13

Renormalized Lagrangian

$$\mathcal{L} = \tfrac{1}{2} Z_3 A^{a\mu}\left(g_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)A^{a\nu} + \frac{1}{2\xi} A^{a\mu}\partial_\mu\partial_\nu A^{a\nu}$$

$$- Z_3 g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \frac{Z_4}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu$$

$$- Z_2' \,\partial^\mu \bar{c}^a \partial_\mu c^a + Z_1' g f^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b$$

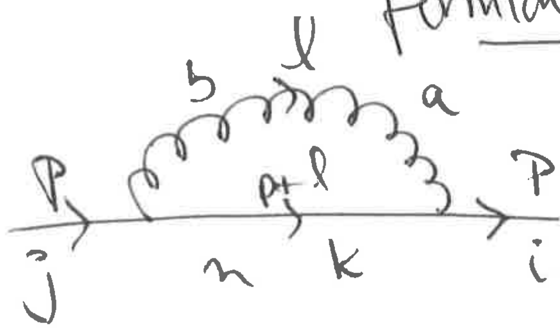
$$+ i Z_2 \bar{\psi}_i \partial\!\!\!/ \psi_i - Z_m m \bar{\psi}_i \psi_i + Z_1 g A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j$$

g appears in several places: expect g.I. will ensure it renormalizes same way in all expressions (Slavnov-Taylor / Ward identities)

Comparing bare with renormalized theories:

$$\boxed{g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \mu^\epsilon}$$

we need to compute vertex, & quark prop and gluon propagator to determine the Z's fermion self energy: (6)



$\Sigma(p)^{ij}_{\alpha\beta}$ ← color, ← spinor

↪ Z_2

same diagram as QED

except lines carry extra color indices ...

$$\sum_a \left(T^a T^a\right)_{ij}$$

since gluon prop is diagonal in color

↑ sum over index a

$$= C_2(R)\,\delta_{ij}$$

thus pole looks like:

$$-\frac{g^2}{8\pi^2\epsilon} C_2(R)\,\delta_{ij}\,\not{p}$$

cancel by

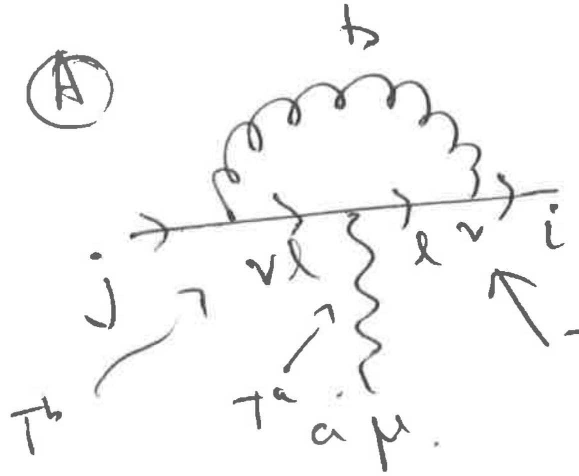
$$-(Z_2-1)\,\delta_{ij}\,\not{p}$$

$\overline{MS}$

ie $Z_2 = 1 - C_2(R)\dfrac{g^2}{8\pi^2}\dfrac{1}{\epsilon} + O(g^4)$

quark-quark gluon vertex $V^{ija}_{\mu}(k)$ (7)

$\leftarrow Z_1$



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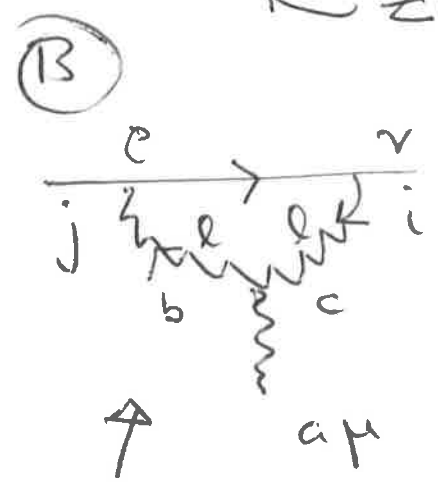


Same as QED except for multiplicative factor

new diagram

$$\sum_{a,b}\left(T^bT^aT^b\right)_{ij} = T^b\left(T^bT^a + if^{abc}T^c\right)$$

$$= C_2(R)T^a + \frac{1}{2}if^{abc}[T^b,T^c]$$

$$= C_2(R)T^a + \frac{1}{2}if^{abc}if^{bcd}T^d$$

$$= C_2(R)T^a - \frac{1}{2}(T^a)^{bc}(T^d)^{cb}T^d$$

$$= C_2(R)T^a - \frac{1}{2}T(A)T^a$$

from QED. get divergent part of this diagram as (8)

$$\left[C_2(R) - \tfrac{1}{2}T(A)\right] \frac{g^2}{8\pi^2 \epsilon}\, i g T^a_{ij} \gamma^\mu$$

↳ hence

$$\delta Z_1 \underset{(A)}{=} - \left[C(R) - \tfrac{1}{2}T(A)\right] \frac{g^2}{8\pi^2} \frac{1}{\epsilon} + O(g^4).$$

Diagram (B) is $\boxed{\text{new}}$

Set external momenta $= 0$ (since div part is indep of them)

Get

$$(ig)^2 g f^{abc} (T^c T^b)_{ij} \left(\frac{1}{i}\right)^3 \times$$

$$\int \frac{d^4 l}{(2\pi)^4} \frac{\gamma_\rho(-\not{l} + m)\gamma_\nu}{l^2 l^2 (l^2+m^2)} \times$$

$$\left[(l - (-l))^\mu g^{\nu\rho} + (-l - 0)^\nu g^{\rho\mu} + (0 - l)^\rho g^{\mu\nu}\right]$$

log divergent — $\frac{1}{\epsilon}$ pole.

Again, color factor is

$$f^{abc}T^cT^b = \frac{1}{2}f^{abc}[T^c,T^b]$$

$$= \frac{1}{2}i f^{abc}f^{cbd}T^d.$$

$$= -\frac{1}{2}iT(A)T^a. \quad \underline{\text{as before}}$$

usual tricks $\Longrightarrow$

$$\delta Z_{1,(13)} = \frac{-3}{2}T(A)\frac{g^2}{8\pi^2}\frac{1}{\epsilon} \times gT^a_{ij}\gamma^\mu.$$

thus

$$Z_1 = 1 - [C_2(R) + \tfrac{T(A)}{2}]\frac{g^2}{8\pi^2}\frac{1}{\epsilon}$$

$$+ \frac{-3}{2}T(A)\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4)$$

ie $$Z_1 = 1 - [C(R) + T(A)]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4)$$

Feynman, $\overline{\text{MS}}$ scheme

vacuum polarization

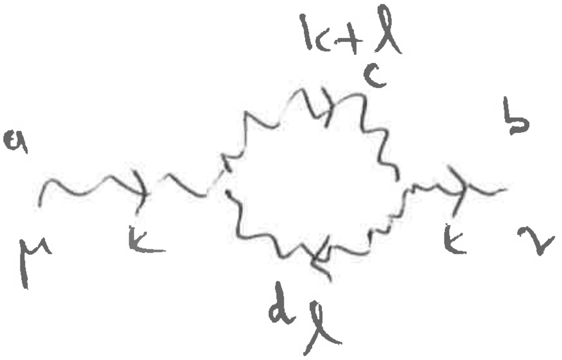
Now for Z_3 :—

$\Pi^{ab}_{\mu\nu}(k)$

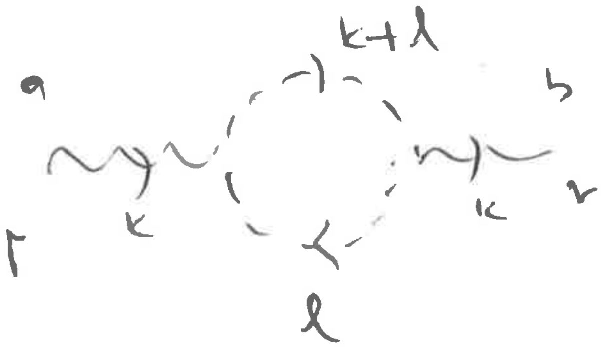
vac. polarization

4 diagrams

①



② gluon.



③ ghost

④ fermions (N_F) rep R.

Diagram ①

naively quadratically div

$$\int d^4 l / l^2 \sim \Lambda^2$$

but pure YM has no scale to compensate Λ (remember that Π is dimensionless once we pull out $(k^2 g^{\mu\nu} - k^\mu k^\nu)$ factor) ← G.I.

thus this term $= 0$ (explicitly true in dim reg)

thus ① ⟹ 0 !

Diagram ④. Same as QED except dressed by color factors

$$\mathrm{Tr}(T^a T^b) = T(R)\,\delta^{ab} \; (\times n_f)$$

ie divergent piece of ④ is

$$\frac{-ig^2}{6\pi^2} n_F T(R) \delta^{ab} \frac{1}{\epsilon} (k^2 g^{\mu\nu} - k^\mu k^\nu)$$

structure inherited from G.I

What about diagrams ② + ③

Log divergent contributions to $\Pi^{ab}_{\mu\nu}$

∴ Expect $1/\epsilon$ poles in dim reg.

Both also have adj fields in loop

→ color factor

$$g^2 f^{acd} f^{bcd} = T(A)\,\delta^{ab}$$

Sum of ② + ③ must (by G.I.) produce

term like $(\underbrace{k^2 g^{\mu\nu} - k^\mu k^\nu}_{= P^{\mu\nu}})$ ← transverse structure.

↓ MS, tricks → div piece

$$\frac{-ig^2}{16\pi^2} T(A)\,\delta^{ab}\,\frac{1}{\epsilon}\,(P_{\mu\nu}) \times \underset{\displaystyle = -10/3}{\text{factor}}$$

$$\therefore\; Z_3 = 1 - \frac{g^2}{6\pi^2} n_F T(R) \frac{1}{\epsilon}$$

$$+ \frac{g^2}{16\pi^2} \frac{10}{3} T(A) \frac{1}{\epsilon}$$

i.e. $$Z_3 = 1 + \frac{g^2}{8\pi^2}\left(\frac{5}{3}T(A) - \frac{4}{3} n_R T(R)\right)\frac{1}{\epsilon} + O(g^4)$$

Now $g_0 = \dfrac{Z_1}{Z_2 Z_3^{1/2}} g \mu^{\epsilon/2}$

$$\begin{cases} Z_1 = 1 - [C_2(R) + T(A)] \dfrac{g^2}{8\pi^2} \dfrac{1}{\epsilon} \\ Z_2 = 1 - C_2(R) \dfrac{g^2}{8\pi^2} \dfrac{1}{\epsilon} \\ Z_3 = 1 + [5/3\, T(A) - 4/3\, n_F T(R)] \dfrac{g^2}{8\pi^2} \dfrac{1}{\epsilon} \end{cases}$$

$$\therefore \ln g_0 = \ln Z_1 - \ln Z_2 - \tfrac{1}{2} \ln Z_3 + \ln g + \epsilon/2 \ln \mu$$

$$\ln g_0 = g^2/8\pi^2 \Big(-C_2(R) + C_2(R) - T(A) - \tfrac{1}{2}\big(5/3\, T(A) - 4/3\, n_F T(R)\big) \frac{1}{\epsilon} + \ln g + \epsilon/2 \ln \mu$$

$$\frac{\partial \ln g_0}{\partial \ln \mu} = 0 \Longrightarrow$$

$$2g^2/16\pi^2 \big(4/3\, n_F T(R) - 11/3\, T(A)\big) \frac{1}{\epsilon} \frac{\partial g}{\partial \ln \mu} + \frac{1}{g} \partial g / \partial \ln \mu + \epsilon/2 = 0$$

$$2g^2/16\pi^2 \left(\tfrac{4}{3} n_F T(R) - \tfrac{11}{3} T(A)\right) \frac{1}{\epsilon} \frac{\partial g}{\partial \ln\mu}$$

$$+ \frac{\partial g}{\partial \ln\mu} + \epsilon g/2 = 0$$

at $O(\epsilon)$

$$\frac{\partial g}{\partial \ln\mu} = -\epsilon g/2 + \cdots$$

<u>set</u>: $$\frac{\partial g}{\partial \ln\mu} = \beta(g) - \epsilon g/2$$

$O(\epsilon^0)$:

$$2g^2/16\pi^2 \left(\tfrac{4}{3} n_F T(R) - \tfrac{11}{3} T(A)\right)\left(-g/2\right)$$

$$+ \beta(g) = 0$$

$$\boxed{\beta(g) = -\frac{g^3}{16\pi^2}\left(\tfrac{11}{3} T(A) - \tfrac{4}{3} n_F T(R)\right)} + O(g^5)$$

asymptotic freedom

$g \to 0$ $\mu \to \infty$ iff $\boxed{\tfrac{11}{3} T(A) - \tfrac{4}{3} n_F T(R) > 0}$

p. theory breaks down

conversely g grows as $\mu \to 0$

<u>confinement?</u>