BRST symmetry

Once the gauge action is (obviously) no longer gauge invariant (G.I)

However we will see that it does possess another (global) symmetry – BRST.

* Using BRST can derive analog of QED Ward identities – called Slavnov-Taylor identities which relate Green functions

→ guarantee that only 4 Z's needed to renormalize quantum theory.

(eg. only covariant derivatives can appear in quantum effective action)

* Also see that ghosts cannot appear as external particles in Feynman diagrams

Toy Model

Consider 1d bosonic model with field $X(t)$

& classical action $S(X) = 0$ (!)

Quantization requires us to define path integral.

$$Z = \int DX \, e^{0}$$

Clearly ∞! Why? because S is invariant under (huge) local symmetry

$$\delta X(t) = s(t) \quad \text{where shift } s(t) \text{ is arbitrary.}$$

How to handle?

Impose gauge condition

$$N(X) - \omega(t) = 0$$

$$Z = \int DX \, \delta(N - \omega) \det\left(\frac{\delta N}{\delta s}\right) e^{0}$$

FP term

note: variation of N under s is just the derivative of N!

As before represent det using "ghosts"

$$\rightarrow Z = \int DX\, Dc\, D\bar{c}\, \delta(N-\omega)\, e^{\int \bar{c} \frac{\partial N}{\partial X} c}$$

As before Z indep of ω $\rightarrow$ integrate over ω

with weight $e^{-\frac{1}{2\alpha}\int \omega^2 dt}$

$$\hookrightarrow Z' = \int DX\, Dc\, D\bar{c}\, e^{\int \bar{c} \frac{\partial N}{\partial X} c - \frac{1}{2\alpha} N^2}$$

(notice: action of this theory is entirely gauge fixing terms!)

Notice it is invariant under fermionic

symmetry

$$QX = c\epsilon$$
$$Qc = 0$$
$$Q\bar{c} = \frac{1}{\alpha} N \epsilon$$

ϵ - infinitesimal anticommuting parameter

a note $Q^2 = 0$ using EOM. $(\bar{c})$

"nilpotent" $\uparrow$ on shell

Notice can rewrite bosonic piece as

$$\frac{1}{\alpha}\left(NB+B^2/2\right)$$

together with integration over B ie

$$Z' = \int DX\,Dc\,D\bar{c}\,DB\; e^{\underbrace{\int \bar{c}\frac{\partial N}{\partial \alpha}c + \frac{NB}{\alpha} + B^2/2\alpha}_{S_{gf}}}$$

$$= \int Dx\,Dc\,D\bar{c}\; e^{\int \bar{c}\frac{\partial N}{\partial \alpha}c - N^2/2\alpha}$$

notice that S_{gf} is invariant under "offshell" version of Q

$$Q\,x = c$$
$$Q\,c = 0$$
$$Q\,\bar{c} = B/\alpha$$
$$Q\,B = 0$$

where $Q^2=0$ without use of EOM

"off-shell symmetry"

Remarkably

$$S_{gf} = Q\int \bar{c}\left(N + \tfrac{1}{2}B\right)dt$$

& mind how to see $QS_{gf}=0$ ...

Notice, any state/op which was initially gauge inv. (now shift invariant) is automatically Q invariant [Q acts like fermionic G. transform]

eg $\int dt \left(\frac{df}{dt}\right) = \int df = \Delta f = 0$ with suitable b.c.s

Notice state with single ghost/antighost must vanish since violates ghost # symmetry of S_{gf} (ie $c, \bar{c}$ opposite U(1) charges)

Consider op with both ghost/antighost

$$\bar{c}\, c\, |\psi_{phys}\rangle$$

this can be written $Q|\phi\rangle$

ie if $|\phi\rangle \sim |\bar{c} f(x)\rangle$

$Q|\phi\rangle \sim k\bar{c}\, f'(x)\rangle + \ldots$

such a state would have zero norm since $Q^2 = 0$ $\langle\phi|Q^\dagger Q|\phi\rangle = 0$

Thus to exclude ghosts from external lines must require that any state of form

$Q|\phi\rangle$ not lie in physical Hilbert space

Technically we say that "physical states lie in cohomology of Q"

i.e. states that are annihilated by Q but cannot be written as $Q|\text{something}\rangle$

<u>Aside</u>

take $N(x) = dx/dt + P'(x)$

$$\int N^2 = \int \left[(dx/dt)^2 + P'^2(x) \right] dt$$

$$\bar{c} \frac{\partial N}{\partial x} c \rightarrow \int \left[\bar{c} \left(d/dt + P''(x) \right) c \right] dt$$

Witten's SUSY QM when physical fermions $\psi, \bar{\psi}$ identified with ghosts $c, \bar{c}$

BRST $\xrightarrow{\text{part of}}$ ($N=2$) supersymmetry

physical states of topological theory $\Longrightarrow$ vacuum of SUSY theory

similar games for $d>1$ — $N \geq 2$

cohomological TQFT $\longrightarrow$ twisted SUSY theories

Back to YM

Require corresponding BRST symmetry

(1) Like infinitesimal gauge transformation when it acts on A_μ, ψ. Thus $Q S_{YM} = 0$ automatically

(2) nilpotent

(3) transform ghost + auxiliary fields B such that

$$S_{gf} = Q \Lambda$$

Thus/

$$Q A_\mu^a = D_\mu^{ab} c^b$$
$$= \partial_\mu c^a - g f^{abc} A_\mu^c c^b$$

$$Q \psi_i = i g c^a (T_R^a)_{ij} \psi_j$$

lets check $Q^2 = 0$ on ψ

$$i g Q c^a T^a_{R ij} \psi_j - i g c^a T^a_{R ij} Q \psi_j$$

minus sign reflects fermionic character of Q ie $Q^2 \psi_i = ig Q c^a T^a_{Rij} \psi_j + g^2 c^a c^b \times$

$$(T^a_R T^b_R)_{ik} \psi_k$$

using $c^a c^b = -c^b c^a$

$$\Rightarrow Q^2 \psi_i = ig\left(Q c^{c} + \tfrac{1}{2} g f^{abc} c^a c^b \right) T^c_{Rij} \psi_j$$

thus $Q^2 = 0$ iff.

$$Q c^c = -\tfrac{1}{2} g f^{abc} c^a c^b$$

<u>homework problem</u> : check $Q^2 A = 0$ also ...

Following toy model assume

$$Q \bar{c}^a = B^a \quad \& \quad Q B^a = 0$$

$\rightarrow Q^2 = 0$ on all fields as required

Assume S_{gf} can be written

$$Q\left(\int \bar{c}^a(x) \left(\tfrac{1}{2\xi} B^a - G^a \right) \right)$$

with $G^a = \partial_\mu A^{\mu a}$ as before.

check

$$-\frac{1}{2\xi} B^{a\,2} + \bar{c}^a Q(\partial_\mu A^{\mu a})$$

$$- B^a G^a$$

Integrate over B as before $\Rightarrow$

$$-\frac{1}{2\xi}(\partial . A)^2 - \partial^\mu \bar{c}^a(x) D_\mu^{ab} c^b$$

$\boxed{\text{correct gauge fixing terms!}}$

Consequences

When we discussed 1 loop structure of YM introduced set of 7 Z's (excluding fermion mass) that determine renormalized theory

BRST symmetry : need only 4!

Z_1, Z_2, Z_3 + single Z associated with $\mathcal{L}_{gf} + \mathcal{L}_{gh}$ i.e $Z' Q[\bar{c}(B-G)]$

Why?

All terms must be BRST symmetric ← ie gauge inv. or Q (something)

Manifestation of this: Slavnov-Taylor identities

$$\langle Q\,O(A, \psi, c, \bar{c}, B)\rangle = 0$$

As for toy model physical states lie in cohomology of Q

ie $Q|\psi_{phys}\rangle = 0$

but $|\psi_{phys}\rangle \neq Q|\chi\rangle$

another way of seeing this: such a state would have vanishing norm.

Furthermore, since $[H, Q] = 0$

states in this class evolve into state of this class

→ use this to show that only transverse polarizations of gluons are physical