Lecture 8

Saw when constructing path integral for free photons sufficient to integrate only over A_μ satisfying Lorentz gauge $\partial^\mu A_\mu = k^\mu A_\mu = 0$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \rightarrow \frac{1}{2} A_\mu k^2 P^{\mu\nu} A_\nu$$

where $P^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ is a <u>projector</u>

which guarantees this since

$$k_\mu P^{\mu\nu} = 0$$

After quantum correction we expect that theory maintains G.I

$\therefore$ this structure must survive in quantum action for A_μ.

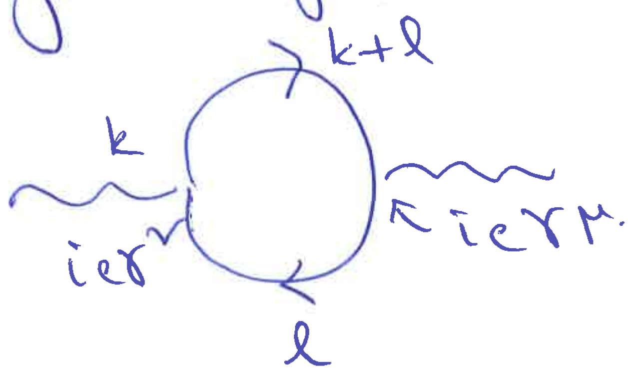
ie $A_\mu \Pi_{\mu\nu} A_\nu$ with $\Pi_{\mu\nu}(k^2) \sim k^2 \Pi(k^2) P_{\mu\nu}$

thus $\Pi^{(2)\,\text{photon}}_{\mu\nu} = P_{\mu\nu} k^2 (1 + \Pi(k^2))$

quantum corrections

& $\Pi(k^2) =$

Feynman graph.



note
conserves momentum at vertex

integrate over loop momenta

$$i\Pi_{\mu\nu}(k) = (-1)(ie)^2\left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left(S(l+k)\gamma^\mu S(l)\gamma^\nu\right)$$

closed fermion loop

charges

sources

move along fermion line & contract spinor indices

& $S(p) = \dfrac{-\not p + m}{p^2 - m^2 + i\epsilon}$

From this point on lets go to Euclidean space

$$S(p) \to \frac{-\not p + m}{p_E^2 + m^2}$$

Trick due to Feynman for combining the terms in the denominator...

$$\frac{1}{(\ell^2+m^2)((\ell+k)^2+m^2)}$$

$$= \int_0^1 dx \left[x((\ell+k)^2+m^2) + (1-x)(\ell^2+m^2) \right]^{-2}$$

$$= \int_0^1 dx \left[\ell^2 + 2x\ell\cdot k + xk^2 + m^2 \right]^{-2}$$

$$= \int_0^1 dx \left[(\ell+xk)^2 + x(1-x)k^2 + m^2 \right]^{-2}$$

$$= \int_0^1 dx \left[q^2 + D^2 \right]^{-2}$$

where $\left.\begin{array}{l} q = \ell + xk \\ D = x(1-x)k^2+m^2 \end{array}\right\}$ & we will replace $\int d^4\ell$ by $\int d^4q$.

$$\therefore i\Pi = -e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{4N^{\mu\nu}}{(q^2+D^2)^2}$$

$$4N^{\mu\nu} = \mathrm{Tr}\left((-\not{\ell} - \not{k} + m)\gamma^\mu (-\not{\ell} + m)\gamma^\nu \right)$$

Using trace properties of γ matrices

$$N^{\mu\nu} = (l+k)^\mu l^\nu + l^\mu (l+k)^\nu$$
$$- [l(l+k) + m^2] g^{\mu\nu}$$

putting $l = q - xk$ & dropping terms linear in q (integrate to zero)

$$\to \; N^{\mu\nu} = 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu$$
$$- (q^2 - x(1-x)k^2 + m^2)\, g^{\mu\nu}$$

using $\int q^\mu q^\nu f(q^2)\, d^4q = \frac{1}{4} g^{\mu\nu} \int d^4q\, q^2 f(q^2)$

one can show that

$$N^{\mu\nu} \to 2x(1-x)\,(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

as expected!

Final trick: the q integral is divergent. Need to regulate it

Choose to make dimension $d < 4$

formulae exist for integrals in general d!

Find

$$\Pi(k^2) = \frac{-e^2}{\pi^2} \int_0^1 dx\, x(1-x)\left[\frac{1}{\epsilon} - \frac{1}{2}\ln\left(D/\mu^2\right)\right]$$

where $\epsilon = 4-d$ – note pole as $\epsilon \to 0$.

Points to note

(*) Naively Π quadratically divergent with momentum cut-off Λ. ~~[illegible]~~
(dimensional analysis)
But since $\Pi_{\mu\nu} \sim k_\mu k_\nu$ in external momenta we see only log divergence in $\Pi(k^2)$

(*) Use dimensional regularization because momentum cut-off Λ breaks gauge invariance.
In low enough dimensions integral is finite
It is always true that log divergences give rise to simple poles in $\epsilon = 4-d$..

2 new questions arise

① How do we take limit $\epsilon \to 0$

② What is μ? Introduce to keep e dimensionless when $d \neq 4$

$$e \to e\mu^{\epsilon/2}$$

Actually we are not done. Expression we have written down ignores quantum corrections parametrized by the Z_i's

Feynman graph

$$\leftarrow i(Z_3 - 1) P_{\mu\nu} k^2$$

Choose Z_3 to cancel off the $1/\epsilon$ pole

$$(Z_3 - 1) = \frac{-e^2}{6\pi^2}\frac{1}{\epsilon} + \text{"finite stuff"}$$

Choose "finite stuff" to remove dependence of $\Pi(k^2)$ on scale μ.

To accomplish this impose physical normalization conditions on $\Pi(k)$

Eg/ Want to keep coeff of $F_{\mu\nu}^2$ equal to $-1/4$ after quantum correction for ~~constant fields~~ $(k \to 0)$ photons on mass shell.

$$\therefore \quad k^2 P^{\mu\nu}(1+\Pi(k^2)) = 1 \quad \text{at } k^2 = 0$$

$$\therefore \quad \boxed{\Pi(0) = 0}$$

Thus $\Pi(0) = -\frac{e^2}{6\pi^2} \ln M^2/\mu^2 + (Z_3 - 1)_{\text{finite}}$

$$\text{ie } (Z_3 - 1) = \frac{-e^2}{6\pi^2}\left(\frac{1}{\epsilon} - \frac{1}{2}\ln\frac{M^2}{\mu^2}\right)$$

This means:

$$\Pi(k^2) = \frac{+e^2}{2\pi^2}\int_0^1 x(1-x)\,\ln\frac{D}{m^2}\,dx$$

$$+O(e^4)$$

(the $\ln m^2/\mu^2$ cancels against the same term in (Z_3-1) correction.)

Notice

Expansion is now finite as $\epsilon \to 0$ & indep of μ!

General feature of the process of renormalization

Perturbative renormalization

* Feynman graphs yield naive ∞'s
* These may be subtracted (cancelled) by using counterterms Z_i (represent the most general interactions consistent with symmetries)

Notice the values of the Z_i are not known a priori

* Once you pay for removing these ∞'s is a dependence on ~~a~~ new scale μ

Physics should not depend on μ.

* Choose <u>normalization conditions</u> which remove dependence on μ. Fixes finite part of Z_i
* <u>Notice</u>. At the end of the day the proper vertices have now developed non-trivial dep on momentum k.