Another look at $\Pi$ Lecture 9

$$\Pi(k^2) = \frac{-e^2}{\pi^2}\int_0^1 dx\, x(1-x) \times \left[\frac{1}{\epsilon} - \frac{1}{2}\ln D/\mu^2\right]$$

$$\therefore \Pi(0) = -e^2/\pi^2 \int_0^1 dx\, x(1-x)\left(\frac{1}{\epsilon} - \frac{1}{2}\ln m^2/\mu^2\right)$$

thus

$$\Pi(k^2) = \Pi(0) + \frac{-e^2}{\pi^2}\int_0^1 dx\, x(1-x)\left[-\frac{1}{2}\ln\frac{D}{\mu^2} + \frac{1}{2}\ln\frac{m^2}{\mu^2}\right]$$

$$= \Pi(0) - \frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\,\frac{1}{2}\ln m^2/D \quad \circledast$$

choose

$$\Pi_{\mu\nu}(k^2) = k^2 P_{\mu\nu}(1+\Pi(k^2)) \to k^2 P_{\mu\nu}$$

as $k^2 \to 0$ (photon on mass shell)

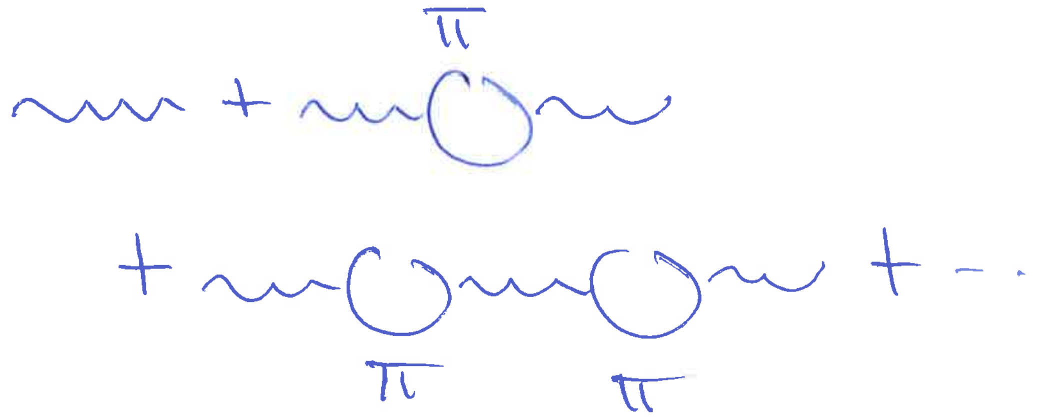
ie $\Pi(0) = 0$ $k^2 = 0$

$\therefore$ consistent with $*$ as before

$$\boxed{\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\ln\frac{D}{m^2}\,dx}$$

# Physical interpretation of $\Pi$

$G^{\mu\nu} =$




$$= \frac{e^2 P^{\mu\nu}}{k^2} + \frac{1}{k^2}\Pi^{\mu\nu}\frac{1}{k^2} + \frac{1}{k^2}\Pi^{\mu\nu}\frac{1}{k^2}\Pi^{\mu\nu}\frac{1}{k^2} + \ldots$$

Since $\Pi^{\mu\nu} = k^2 P_{\mu\nu} \cdot \Pi$

$$= \frac{e^2 P^{\mu\nu}}{k^2}\left(1 + \Pi + \Pi^2 + \ldots\right)$$

$$= \frac{e^2 P^{\mu\nu}}{k^2}\,\frac{1}{(1-\Pi(k))}.$$

thus Fourier transform of $V(r)$ yields

effective charge

$$V_{eff}(p) = \frac{e^2_{eff} P^{\mu\nu}}{p^2}$$

ie $e^2_{eff}(p) = \dfrac{e^2}{1-\Pi(p)}$ Landau $\leftarrow$ pole!

breakdown of p. theory... at high energies

$$\frac{1}{e^2_{eff}(p)} = \frac{1}{e^2} - \frac{\Pi(p)}{e^2}$$

$\leftarrow \frac{1}{2\pi}\int_0^1 x(1-x)\,dx \ln \frac{D}{m^2}\,dx$

for $k^2 \gg m^2$

$$\frac{1}{e^2_{eff}} = \frac{1}{e^2} - \frac{1}{12\pi^2}\ln k^2/m^2.$$

running coupling constant!

notice running switches off if $k^2 < m^2$.

$k^2 \to \infty$ $\quad \frac{1}{e_{ff}^2}$ decreases

ie $e_{ff}^2$ increases at short distance

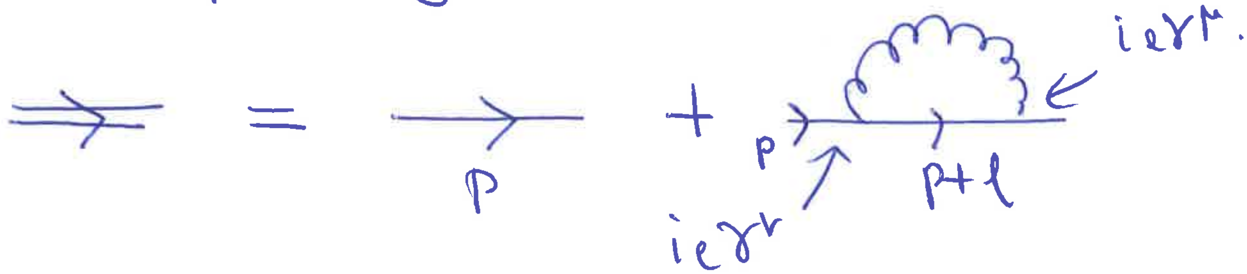
conversely

$e_{ff}^2$ decreases in IR.

# Fermion Self Energy

$$\Gamma^{(2)\,\text{fermion}} = S_{\text{full}}(p)^{-1} = \not{p} + m - \Sigma(p)$$

corresponding to




$$i\Sigma(p) = (ie)^2 \left(\frac{1}{i}\right)^2 \int \frac{d^4 l}{(2\pi)^4} \gamma^\nu S(p+l) \gamma^\mu \Delta_{\mu\nu}(l)$$

* note: $\Sigma$ has a matrix structure (spinor indices)

contract over spacetime indices $\mu, \nu$

* It has a naive linear divergence

$\int \frac{d^4 p}{p^3}$ large $p$ but expect that

$\Sigma \sim \not{p} + \dots$ so only $\log \Lambda$ divergence

(also integrand odd function of $p$)

$\longrightarrow \frac{1}{\epsilon}$ pole expected in dim reg.

* put in photon mass $m_\gamma$ to regulate IR.

Again use dim reg to render the integral finite

After some work (in renormalized p. theory)

$$\Sigma(\not{p}) = \frac{-e^2}{8\pi^2}\int_0^1 dx \left[(2-\epsilon)(1-x)\not{p} + (4-\epsilon)m\right] \times \left[\frac{1}{\epsilon} - \frac{1}{2}\ln D/\mu^2\right]$$

$$-(Z_2-1)\not{p} - (Z_m-1)m + O(e^4)$$

$$D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$$

finiteness requires

$$\left.\begin{aligned} Z_2 &= 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right) \\ Z_m &= 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\epsilon} + \text{finite}'\right) \end{aligned}\right\}$$

ie equate poles in $1/\epsilon$ for $\not{p}$ & $m$ separately

to fix the finite terms we again impose normalization conditions on $\Sigma$

Choose on-shell scheme where

$$\Sigma(-m) = 0 \quad + \quad \Sigma'(-m) = 0$$

puts pole of propagator at m

sets coeff of $\bar{\psi}\not{\partial}\psi = 1$ (residue of pole)

To fix $\Sigma(\not p) = 0$ at $\not p = -m$,

forces

$$\Sigma(\not p) = \frac{e^2}{8\pi^2}\left[\int_0^1 dx\,((1-x)\not p + 2m)\ln D/D_0 + K_2(\not p + m)\right] + O(e^4)$$

when $D_0 = D$ evaluated at $\not p = -m$
ie on mass shell

$K_2$ determined by requiring

$$\Sigma'(-m) = 0$$

$$\Rightarrow \quad K_2 = -2\ln(m/m_\gamma) + 1$$

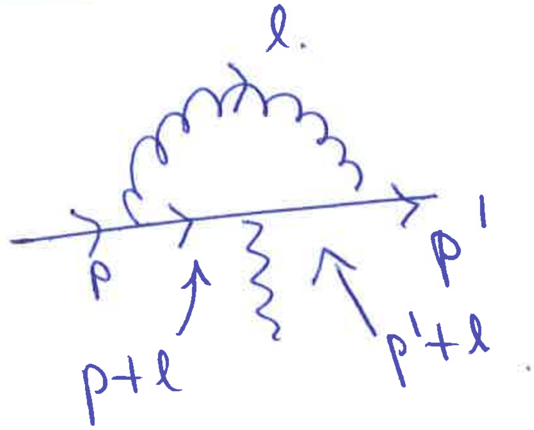
again final $\Sigma$ is finite as $\epsilon \to 0$
and independent of $\mu$.

[ to remove $m_\gamma$ dependence we need first
to sum over low energy photons not resolved
in any real experiment & then take limit $m_\gamma \to 0$
— not today! ]

# Vertex correction

$\Gamma^{\bar{\Psi}\Psi A} \equiv V^{(3)}$

given by




$$V^{(3)} = (ie)^3 \left(\frac{1}{i}\right)^3 \int \frac{d^4 l}{(2\pi)^4} \Big[\gamma^\rho S(p+l)\gamma^\mu S(p'+l) \gamma^\nu \Delta_{\nu\rho}(l)\Big]$$

naive log divergence $\rightarrow \frac{1}{\epsilon}$ pole dim reg

subtract by adding a contribution from counterterm $(Z_1 - 1)$

divergent part of $Z_1 - 1$ cancels $1/\epsilon$ pole

$$\Rightarrow \quad Z_1 - 1 = \frac{-e^2}{8\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)$$

Finite pieces again determined by using on-shell scheme where

$V^\mu = ie\gamma^\mu$ at $p^2 = p'^2 = -m^2 \quad p_\gamma^2 = 0$

$\Downarrow$ physical electron charge.

## Points to note

① $Z_1^{div} = Z_2^{div}$

Turns out this is true also for finite pieces to all orders in e!

ie $\boxed{Z_1 = Z_2}$ always

Ward identity (like $k^\mu \Pi_{\mu\nu} = 0$)

Consequence of gauge invariance

Allows all expressions to be rewritten (operators)

in terms of covariant derivatives

even in quantum action

relates $\bar{\psi}(a_1 \not{\partial} + b_1 e \not{A})\psi$ ie $\boxed{a_1 = b_1}$ always

Thus only 3 inputs needed to renormalize QED

$Z_1, Z_3, Z_m$

# Renormalization Group (continuum)

So far written [indep of physical quantities on scale $\mu$]

$$\mathcal{L} = \mathcal{L}_0 + Z_1 e \bar{\psi} \not{A} \psi + i(Z_2 - 1)\bar{\psi} \not{\partial} \psi$$
$$- (Z_m - 1) m \bar{\psi}\psi - \tfrac{1}{4}(Z_3 - 1) F_{\mu\nu}F^{\mu\nu}$$

with $Z$'s chosen to cancel off divergences in Feynman diagrams with loops

+ $\mu$ dependent finite parts chosen to satisfy physical normalization condition

↑ in MS scheme set finite parts to zero

Can write this as:

$$\mathcal{L} = -\tfrac{1}{4} F^0_{\mu\nu} F^{\mu\nu}_0 + e_0 \bar{\psi}_0 \not{A}_0 - m_0 \bar{\psi}_0 \psi_0$$
$$+ i \bar{\psi}_0 \not{\partial} \psi_0$$

$e_0, m_0, \psi_0, A_0$ bare quantities

They are indep of $\mu$.

yield physical ($\mu$ indep) scattering amplitudes easy to relate the two:

$$A_0 = \sqrt{Z_3}\, A$$
$$\psi_0 = \sqrt{Z_2}\, \psi$$
$$m_0 = Z_m / Z_2 \; m$$
$$e_0 = \left( Z_1 / Z_2 \sqrt{Z_3} \right) e \mu^{\epsilon/2}$$

↑ again this renders e dimensionless when $d \neq 4$

aside on dimensional analysis

$\int \bar{\psi} \not{\partial} \psi$ must be dimensionless

$$[\psi] = a^{\frac{1}{2}(1-d)}$$

similarly $\int F^2 \rightarrow [A] = a^{\frac{1}{2}(2-d)}$

∴ vertex $\int \bar{\psi}\psi A$ dimension $a^{2+d - 3/2\, d} = a^{1/2(4-d)}$

writing $d = 4 - \epsilon \longrightarrow \underline{\underline{a^{1/2\,\epsilon}}}$

therefore coupling for vertex goes as $a^{-\epsilon/2}$

∴ $e \Rightarrow \mu^{\epsilon/2}\, e$ ← dimensionless

$\mu$ is energy scale $\sim 1/a$