Anomalies

In general, presence of classical symmetry leads to Ward identities in quantum theory

eg. Slavnov–Taylor identities in YM

$$\langle \delta O \rangle = 0$$

ex of more general expression when $\delta O = 0$

eg consider action invariant under continuous symmetry parametrized by α

eg $\int \bar{\psi}(i\not{\partial} + m)\psi$ $\quad \left.\begin{array}{l}\psi \to e^{-i\alpha}\\ \bar{\psi} \to e^{+i\alpha}\end{array}\right\}$

consider $\alpha \to \alpha(x)$

$$\delta S = \int \bar{\psi}\gamma_\mu \psi \, \partial_\mu \alpha \qquad \text{(Noether thm)}$$

classical invariance $\to$ $\partial_\mu J_\mu = 0$ $\quad J_\mu = \bar{\psi}\gamma_\mu\psi$

quantum theory:

$$\langle O \rangle = \langle O' \rangle = \int (O + \delta O)\, e^{-S - \delta S}$$

ie $\langle O \delta S \rangle = 0$

ie $\langle O \partial_\mu J^\mu \rangle = 0$

can insert $\partial \cdot J$ under correlation function & get zero

Aside/ Note Dirac action also invariant under chiral symmetry

$$\left.\begin{array}{l} \psi \to e^{i\alpha\gamma_5} \\ \bar\psi \to e^{i\alpha\gamma_5} \end{array}\right\}$$ except for mass term

since $[\gamma_5, \gamma_\mu]_+ = 0$

$$\partial_\mu J^{\mu 5} = 2im\bar\psi\gamma_5\psi$$

where $J_\mu^5 = \bar\psi\gamma^\mu\gamma^5\psi$

thus in classical theory axial current $J_{\mu 5}$ also conserved if $m=0$

Quantum mechanically it was observed that this is no longer the case

— historical discovery of anomalies

$\boxed{\text{classical symmetry does not survive quantization}}$

Lets see how this works...

Consider

$$Z(A) = \int D\psi D\bar{\psi}\, e^{iS(A)}$$

with $S(A) = \int d^4x\, \bar{\psi}\, i \not{D} \psi$

$D_\mu = \partial_\mu - igA_\mu$ ← background field

Action is manifestly axial invariant. What about $Z(A)$?

Need to look closely at measure $D\psi D\bar{\psi}$

axial/chiral transformation:

$$\psi \to \psi' = (1 - i\alpha\gamma_5)\psi$$

$$\bar{\psi} \to \bar{\psi}' = (1 - i\alpha\gamma_5)\bar{\psi}$$

α infinitesimal

Expand $\psi, \bar{\psi}$ on basis of eigenfunctions of $\not{D}$

$\not{D} v_i = \lambda v_i$ where $\int d^4x\, v_i^\dagger v_j = \delta_{ij}$

$$\psi = \sum_i v_i(x)\, \alpha_i \quad , \quad \bar{\psi} = \sum v_i(x)\, \beta_i$$

$$D\psi D\bar\psi = \prod_i d\alpha_i \, d\beta_i$$

& under axial transf.

$$\alpha_j' = \alpha_j \delta_{ij} + \int d^4x \sum_i v_j^+ -i\alpha\gamma_5 v_i \alpha_i$$

[set $\psi' = \sum v_j \alpha_j'$ & use orthogonality of v_i]

thus get Jacobian $\alpha \to \alpha'$ (similar for β)

$$J_{ij} = \frac{\partial \alpha_j'}{\partial \alpha_i} \qquad \det(J)$$

$$J_{ij} = \delta_{ij} - i\int d^4x \, \alpha \, v_i^+ \gamma_5 v_j$$

implicit sums over group & spinor indices..

$\beta + \alpha$ Jacobian

$$\therefore \det{}^{-2} J = \exp\left(-2\,\mathrm{tr}\ln\left(\delta_{ij} - i\int \alpha v_i^+ \gamma_5 v_j\right)\right)$$

$$= \exp\left(2i\,\mathrm{tr}\int \alpha v_i^+ \gamma_5 v_j\right)$$

now, naively every mode v_i with eigenvalue $\gamma_5 = +1$ comes paired with another with $\gamma_5 = -1$

trivial proof

$$\not{D} v_i = \lambda_i v_i$$

$$\therefore \gamma_5 \not{D} v_i = \lambda_i \gamma_5 v_i$$

$$\therefore \not{D}(\gamma_5 v_i) = -\lambda_i (\gamma_5 v_i)$$

thus ~~$v_i^\dagger \gamma_5 v_i$ ...~~

$$v_i = \tfrac{1}{2}(1+\gamma_5) v_i + \tfrac{1}{2}(1-\gamma_5) v_i$$

$$\gamma_5 v_i = 1 v_i^+ - 1 \cdot v_i^-$$

$$v_i^\dagger \gamma_5 v_i \Rightarrow v_i^{+\dagger} v_i^+ - v_i^{-\dagger} v_i^- = 0.$$

however this proof fails if $\lambda = 0$ ← no need for pairing

Another weakness of this argument is the fact that all the sums $\sum_i$ are infinite! We really should regularize this computation before drawing any conclusions...

One simple way:

$$\exp\left(2i\theta \int d^4x \, \alpha \sum_i v_i^\dagger e^{-\lambda_i^2/M^2} v_i\right)$$

$M \to \infty$ reduces to previous exp

$$\det^{-2} J = \exp\left(2i \lim_{M\to\infty} \alpha \int d^4x\, v_i^\dagger \gamma_5 e^{-\not{D}^2/M^2} v_i\right)$$

now $\not{D}^2 = \gamma_\mu D_\mu \gamma_\nu D_\nu$

$$= D^2 + ig/4\, [\gamma_\mu, \gamma_\nu] F_{\mu\nu}$$

only non-trivial contribution comes from

$D^2 = k^2 \sim M^2$

$$\therefore \det^{-2} C = \exp\left(2i\alpha \lim_{M\to\infty} \langle x| \operatorname{tr}(\gamma_5 e^{-\not{D}^2/M^2}) |x\rangle\right)$$

$$= \lim_{M\to\infty} \alpha\, \operatorname{tr}\left(\gamma_5 \frac{1}{2}\left(\frac{ig}{2M^2}\gamma_\mu\gamma_\nu F_{\mu\nu}\right)^2\right)$$

$$\times \langle x| e^{-\partial^2/M^2} |x\rangle$$

first G.I contribution ...

now $\operatorname{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda) = 4i\,\epsilon_{\mu\nu\rho\lambda}$

$$\langle x| e^{-\partial^2/M^2} |x\rangle = i\int \frac{d^4k}{(2\pi)^4} e^{-k^2/M^2} = iM^4/16\pi^2$$

$$\therefore \det^{-2} C = \exp\left(\frac{-ig^2\alpha(\cancel{M^4})}{16\pi^2 \cancel{M^4}}\,\epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}\right)$$

indep of M!

Note: expanding to higher powers in $A_\mu \to 0$ as $M \to \infty$

Back to $Z \Rightarrow$

$$Z = \int D\psi D\bar{\psi} \exp\left(iS(A) + i\int \alpha \left[\partial_\mu J^{\mu 5} + \frac{g^2}{8\pi^2} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]\right)$$

where dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda}$

i.e. current no longer conserved eg at $m=0$

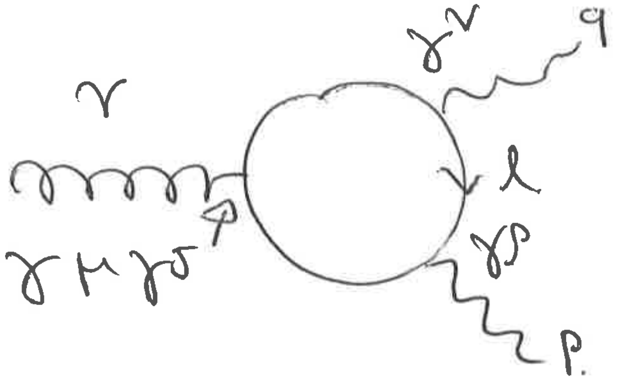
$$\partial_\mu J^{\mu 5} = -g^2/16\pi^2 \, \epsilon_{\mu\nu\rho\lambda} F^{\mu\nu} F^{\rho\lambda}$$

Furthermore we can now argue that the RHS will be non-zero only when a zero mode of fermion appears! ← topological field configs important — later ...

Learnt that anomalies appear when $\int F\tilde{F} \neq 0$ path integral measure not invariant under the symmetry

It is also clear that U.V effects play crucial role (remember that integral was cut-off for $k \sim M$ as $M \to \infty$)

Indeed historically anomaly was first discovered in 1 loop Feynman diagrams



amplitude to create 2 photons from an axial current

$$\langle A_\nu \, j^{\alpha 5} A_\rho \rangle = \Gamma^{\nu\alpha\rho}$$

<u>classically</u> $\langle A_\nu \, \partial_\alpha j^{\alpha 5} A_\rho \rangle = 0$

$\uparrow \bar{\psi}\gamma^5\psi$ <u>pion</u>

$\pi \to 2\gamma$

<u>observed</u> <u>non-zero</u>!

$$\Gamma^{\nu\alpha\rho} \sim \int \frac{\mathrm{tr}\left(\gamma_5 \gamma^\mu (\not{l}-\not{p}) \gamma^\rho \not{l} \gamma^\nu (\not{l}+\not{q})\right)}{(l-p)^2 \, l^2 \, (l+q)^2} \frac{d^4 l}{(2\pi)^4}$$

integral <u>linearly</u> divergent!

Conservation of current

$$\partial_\mu J^{\mu 5} = 0$$

$$\hookrightarrow \quad r_\mu \Gamma^{\mu\rho\nu} = 0$$

which is formally true if ignore divergence..

One way to regulate divergence is to use

Pauli-Villars regularization

$$\frac{\not p}{p^2} \rightarrow \frac{\not p}{p^2} - \frac{(\not p - M)}{p^2+M^2}$$ modify propagator.

$M \to \infty$ no change

but $p \gg M$ propagator damps out faster than

$1/p^2$ → integral convergent

But/ mass M breaks axial symmetry

this effect doesn't disappear as $M \to \infty$

find instead

$$\int \frac{d^4 k}{(2\pi)^4}\, r_\mu I^{\mu\rho\lambda} = g^2/4\pi^2 \, \epsilon_{\alpha\beta\rho\lambda} P_\alpha q_\beta + \dots$$

c.f. $\epsilon_{\alpha\beta\rho\lambda} \partial_\alpha A_\rho \partial_\beta A_\lambda$ again!

General feature: impossible to maintain exact chiral symmetry in q. theory

regulator breaks it, & this effect survives even when regulator is removed

In the case of global axial symmetries like the case discussed this is no big deal

Just means that symmetry is broken & yields interesting phenomena like $\pi^0 \to 2\gamma$ or mass of η' meson in QCD.

↑ not a Goldstone boson since associated to a $U(1)_A$ broken by quantum effects.

<u>However</u>, anomalies pose <u>severe</u> problem for currents associated to <u>gauge</u> symmetries