General comments

Want to evaluate

$$Z = \int D\Phi \, e^{-S(\Phi)/\hbar}$$

Loop expansion ($\hbar$) is saddle pt expansion about constant fields φ.

Other solutions to EOM ~~are~~ exist when field φ is not constant

Typically action for such configs > 0 because of gradient terms

— suppressed relative to perturbative vacuum

But if can arrange for S to be finite they can play a role at strong coupling.

Furthermore such configs are often topologically stable

Topological Aspects Gauge Theories

Consider 1+1 D scalar field theory

$$\mathcal{L} = +\frac{1}{2}\partial^\mu \phi\, \partial_\mu \phi - V(\phi)$$

with $V(\phi) = \frac{1}{8}\lambda(\phi^2 - v^2)^2$

2 ground states $\phi = \pm v$.

In terms of $\delta\phi = \phi - v$ say find mass of particle is $2\sqrt{\frac{1}{8}\lambda 2 v^2} = \underline{\sqrt{\lambda}\, v}$

consider time indep solns to classical field eqns

topology of spatial boundary $= S^0$ (2 pts)

topology of vacuum manifold also 2pts ($\pm v$)

Imagine state where
$$\left.\begin{array}{ll} \phi \to +v & x \to \infty \\ \phi \to -v & x \to -\infty \end{array}\right\}$$

somewhere field must be out of vacuum

→ energy needed

example of a soliton "kink" soln

$$E = \int dx \left(\underbrace{\tfrac{1}{2}\dot{\varphi}^2}_{=0} + \tfrac{1}{2}\varphi'^2 + V(\varphi) \right)$$

$$= \int_{-\infty}^{\infty} dx \left[\tfrac{1}{2}\left(\varphi' \pm \sqrt{2V} \right)^2 \mp \sqrt{2V}\,\varphi' \right]$$

$$= \int_{-\infty}^{\infty} dx\, \tfrac{1}{2}\left(\varphi' \pm \sqrt{2V} \right)^2 \mp \int_{-v}^{v} \sqrt{2V}\, d\varphi$$

$$\int_{-v}^{v} \tfrac{1}{2}\sqrt{\lambda}\,(\varphi^2 - v^2)\, d\varphi$$

$$= \tfrac{1}{2}\sqrt{\lambda}\left[\varphi^3/3 - v^2\varphi \right]_{-v}^{v}$$

$$= \tfrac{1}{2}\sqrt{\lambda}\left(v^3/3 - v^3 \right) \times 2$$

$$= \tfrac{2}{3}\sqrt{\lambda}\, v\, v^2$$

$$= \tfrac{2}{3} m \left(m^2/\lambda \right)$$

Thus minimum E is

$$\tfrac{2}{3} m \left(m^2/\lambda \right)$$

if $\lambda \ll m^2$ weakly coupled

this is large

1st term ≥ 0

thus at minimum energy

$$\frac{d\varphi}{dx} = -\tfrac{1}{2}\sqrt{\lambda}\,(\varphi^2 - v^2)$$

$$\hookrightarrow \int \frac{d\varphi}{\varphi^2 - v^2} = \sqrt{\lambda}/2 \int dx \quad \text{i.e.} \quad \int \frac{d\varphi/v}{(\varphi/v)^2 - 1} = \frac{v\sqrt{\lambda}}{2}(x - x_0)$$

thus

$$\varphi(x) = v \tanh \tfrac{1}{2} m (x - x_0) \qquad \text{kink}$$

stationary, localized finite energy ($E \to 0$ exp fast $|x-x_0|$ large)
soln to classical EOM.

soliton

Notice can get other solutions by Lorentz trans

$$\varphi(x,t) = v \tanh\left(\tfrac{1}{2}\gamma m \left(x - x_0 - \gamma\beta t\right)\right)$$

$$\gamma = (1-\beta^2)^{-1/2}$$

with $E = \gamma \underbrace{\tfrac{2}{3} m \left(m^2/\lambda\right)}_{M}$

P energy of soliton "at rest"

$$= (p^2 + M^2)^{1/2} \quad (\text{exercise from } dp = \gamma E)$$

behaves like massive ptcle --

non-perturbative in character ($1/\lambda$)

in 2 spatial D $\Rightarrow$ domain wall (∞ in 2nd dir)

Notice kink cannot decay

to elementary excitations since requires

∞ energy to change $\phi(-\infty)$ from say

$-v$ to $+v$.

formally can define current

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu}\partial_\nu\phi$$

$\partial_\mu J^\mu = 0$ by antisymmetry of $\epsilon^{\mu\nu}$

but $\int \partial_0 J^0\, dx = -\int \partial_1 J^1\, dx$

i.e. $\frac{\partial}{\partial t} Q \quad = +\frac{1}{2}\int \frac{\partial}{\partial x}\frac{\partial\phi}{\partial t}$

or $Q = \frac{1}{2}\Delta\phi\Big|_{-\infty}^{\infty}$ Q conserved

$$\boxed{Q = v}$$

actually can redefine J^μ to be $\frac{1}{2v}\epsilon\partial\phi$

$\hookrightarrow Q = 1$ for kink

($Q=0$ for trivial vac $x\to\pm\infty$ $\phi\to+v$)

note $\frac{\partial Q}{\partial t} = 0$ conservation of topology ...

Going back to solution

for kink see that

$$E_{min}(M) = \sqrt{\lambda}\,\frac{2}{3}v^2\,\Delta\phi\Big|_{-\infty}^{\infty}$$

$$= \frac{2}{3}\sqrt{\lambda}\,v^3\,Q$$

in general $M \geq Q$

measured in units of $\frac{2}{3}\sqrt{\lambda}v^3 = \frac{2}{3}m\left(\frac{m^2}{\lambda}\right)$

Bogomolnyi bound on mass of

soliton (BPS)

④

Having found sol'n localized in 1D (~~spatial~~ spatial)

look for localized solutions in <u>2D</u>

By analogy need topology of vacuum manifold to be S^1 (to match S^1 spatial boundary topology)

hence try complex scalar field with potential

$$V(\phi) = \tfrac{1}{4}\lambda(\phi^+\phi - v^2)^2$$

vacua: $\phi = v e^{i\alpha}$

analog of kink will be solution that gives

$\alpha(\phi)$

↑ angle specifying pt on <u>spatial boundary</u>

↑ $\phi = 0 \ldots 2\pi$ of spatial S^1

⇒ map from $S^1 \to S^1$

$\boxed{\text{require}}$ $\alpha(\phi + 2\pi) = \alpha(\phi) + 2\pi n$

<u>periodicity</u>

↑ winding #

eg $\phi|_{\text{boundary}} = U(\phi) = e^{in\phi}$

$n=0$ trivial map $\alpha = $ constant
indep of ϕ

$n=1$ simple vortex solution... ($n=-1$ winds opposite way)

In general can have any smooth deformation of $e^{in\phi}$ as map eg $U(\phi)$

(soln at ∞ for φ-vac)

$$n = \frac{i}{2\pi} \int_0^{2\pi} d\phi \, U \partial_\phi U^\dagger \quad \text{winding \#}$$

~~What~~ Are there finite energy solutions to this boundary condition?

try

$$\varphi(r,\phi) = v f(r) e^{in\phi}$$

with $f(\infty)=1$ $\quad f(0)=0$ so $\nabla\varphi$ reg at $r=0$

$$\nabla\varphi = v\left(f'(r)\hat{r} + in r^{-1} f(r)\hat{\phi}\right)e^{in\phi}$$

so $|\nabla\varphi|^2 = v^2\left(f'^2 + n^2 r^{-2} f^2\right)$

$\propto 1/r^2 \quad r\to\infty$

$\therefore$ Energy $\int r\,dr/r^2 \sim \log.$ div.
$r \to \infty$.

so no!

(Derrick's thm)

Abelian Higgs model

How to fix?

Add gauge fields for the global $U(1)$

$$\mathcal{L} = D_\mu\phi D^\mu\phi^\dagger - V(\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$$

On vacuum manifold $\langle\phi\rangle \neq 0$ ← gauge sym broken (more later)

$$|D\phi|^2 = |(\nabla - ieA)\phi|^2$$

choose A so that cancel off this bad large r behavior ....

ansatz for soln now looks like gauge transformation by $e^{in\phi}$ on $\left.\begin{matrix}\phi = v\\ A = 0\end{matrix}\right\}$

gauge transformation of A will now cancel off this term (A=0 initially)

$$\lim_{r\to\infty} A(r,\phi) = \frac{i}{e} e^{in\phi} \nabla e^{-in\phi}$$

ie $\delta A_r = \frac{i}{e} U \partial_r U^{\dagger}$

$$= \frac{n}{er} \hat{\phi}$$

$\therefore$ $(D - ieA)\phi \sim f'(r)\,\hat{r}$ only $\to 0$

as $r \to \infty$

ie can arrange vacuum state annihilated by ∇_{cov}

$$(\nabla - ieA)\, v e^{in\phi} = 0$$

using just gauge invariance

Notice for $n \neq 0$ gauge transformation is large – it cannot be continuously deformed to $U = 1$

implies cannot extend it from $r = \infty$ into interior without finding pt where U is ill-defined (r=0 often) Near the distortion fields must deviate from gauge transformation of vacuum

$\to$ finite energy

This deviation costs energy but it is finite

ansatz for such solution is

$$\phi(r,\phi) = v f(r) U(\phi)$$
$$A(r,\phi) = \frac{i}{e} a(r) U(\phi) \nabla U^{\dagger}(\phi)$$

where $U(\phi) = e^{in\phi}$

and $f(\infty) = a(\infty) = 1$

(needed to approach vac soln)

+ $f(0) = a(0) = 0$ at $r=0$ ($A, D\phi$ well defined)

when $n=1$ — Nielsen-Olesen vortex

non-zero $A \to B$ field (no E - time indep)

$B = \nabla \times A$ flux of B $\Phi = \int B \cdot dS$

$= \oint A \cdot ds$ using Stokes thm

$$\Phi \sim \frac{i}{e} \lim_{r\to\infty} a(r) \int_0^{2\pi} d\phi\, U \partial_\phi U^{\dagger}$$

winding #

therefore

$$\Phi = \frac{2\pi n}{e}$$ flux is quantized

vortex carries magnetic flux inversely proportional to charge!

(type II superconductors)

Solve (numerically) EOM to find

$f(r)$, $a(r)$ for such a vortex

Again can prove a BPS bound:

$$E \geq 2\pi v^2 |n|$$

solution with winding # n can break up into

n solutions with winding # 1

In 3D $\rightarrow$ Nielsen-Olsen string ↳ in extra dimension

(cosmic strings like this)