Lecture 18

Recap: Steepest descent calculations of path I lead us to consider contribution of non-trivial solutions of classical EOM.

eg kinks & Nielsen-Olesen vortices in (1+1) & (2+1) D respectively

Key features of these solutions:

* topological stability requires that topology of vacuum states matches topology of boundary of space eg $S^1 \to S^1$ maps

* Finite energy / ~~localized~~ requires localization yields strong constraints on possibilities eg need gauge field in (2+1) D case

↑ solution must approach vacuum at ∞

* These configs become important at strong coupling

* Typically satisfy BPS bound $E \geq \alpha Q$.

't Hooft-Polyakov monopole

Take scalar field model & lift to (3+1)D

boundary of space now S^2

need vacuum manifold with this topology

eg/ 3 scalar fields frozen to surface of sphere.

to ensure kinetic terms yield finite contribution this should be gauge theory

eg/ $\mathcal{L} = -\frac{1}{2} D^\mu \phi D_\mu \phi - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

& take $V(\phi) = \frac{1}{8}\lambda\left(\phi^a\phi^a - v^2\right)^2$

natural to take ϕ^a to live in adjoint rep of SO(2)

eg $D_\mu^{ab}\phi^b = \partial_\mu\phi^a + e\,\epsilon^{abc}A_\mu^b\phi^c$

vacuum: $\phi^a = v\,\hat{\phi}^a$

$\uparrow$ unit vector

Can set $V=0$ as $r \to \infty$ by

taking $\varphi^a = v\, x^a/r$

[note weird mixing of spacetime/internal indices

— this configuration corresponds to one

where internal direction of ϕ matches

external direction]

Anyhow, to put $D_i \varphi \sim 0$ as $r \to \infty$

need to choose

$$-e\epsilon^{abc}A^b_i\,\varphi^c \sim \partial_i \varphi^a$$

sol= $A^b_i \sim \frac{1}{e}\epsilon^{bij}\frac{x^j}{r^2}$

notice $A_i \sim \frac{1}{r}$ ie $B_i \sim \frac{1}{r^2}$ magnetic field corresponding to this A

magnetic monopole!

$$B \sim \frac{g}{4\pi r^2}\hat{r}.$$

Note that (like vortex) generic pt on vacuum manifold $\phi^a \sim v\,\delta^{3a}$ breaks gauge symmetry down from SU(2) to U(1)

↑
ie can rotate just about
3 direction

thus this magnetic field will be associated with this unbroken (EM) U(1) field

Example of a more general phenomenon:
't Hooft & Polyakov showed that magnetic monopoles arise quite generally when a gauge theory is spontaneously broken (Higgsed) ← removing these in GUT theories was the original motivation for inflation ...

Again, can derive bound a mass of this soliton (here monopole) by following argument

$$M = \int d^3x \left(\tfrac{1}{4} F_{ij}^2 + \tfrac{1}{2} (D_i \phi)^2 + V(\phi) \right)$$

$$= \int d^3x \left(\tfrac{1}{4} [F_{ij} \pm \epsilon_{ijk} D_k \phi]^2 + V(\phi) \mp \tfrac{1}{2} \epsilon_{ijk} F_{ij} D_k \phi^{\nwarrow \to 0} \right.$$

≥ 0 as before

$$\text{ie } M \geq \frac{1}{2} \overbrace{\int d^3x\, \epsilon_{ijk} F_{ij} D_k \phi}^{A}.$$

$$\uparrow$$

3-form topological invariant

depends only on boundary.

furthermore (A) can be written ← Bianchi identity

$$= \frac{1}{2} \int d^3x\, \epsilon_{ijk} D_k (F_{ij} \phi)$$

$$= \frac{1}{2} \int d^3x\, \epsilon_{ijk} \partial_k (F_{ij} \cdot \phi) \quad \left. \begin{matrix} A \to 0 \\ r \to \infty \end{matrix} \right\}$$

but $B_k = \frac{1}{2} \epsilon_{ijk} F_{ij}^a$

Therefore $M \geqslant \int d^3x\, \partial_k (B^a_k \phi^a)$

$$\geqslant v \cdot \int dS.B$$

↑ flux $= g$

(like for $\nabla \cdot E = \rho$)

but clearly $A \sim \frac{1}{e}$

consistent with Dirac's quantization condition $eg = 2\pi n$ ie $g = \frac{2\pi n}{e}$

n arising here as winding # of map

$S^2 \to S^2$ associated with scalar field

hedgehog

$$n = \frac{1}{8\pi} \int d^2\theta\, \epsilon^{abc} \epsilon^{ij} \hat{\phi}^a \partial_i \hat{\phi}^b \partial_j \hat{\phi}^c$$

↑ integral of 2 form

topological invariant!

The mass bound is saturated on configs satisfying

$$F_{ij} = \pm \epsilon_{ijk} D_k \varphi$$

or $B_i = -D_i \varphi$

solutions of this eq^{n} are known as Bogomolnyi-Prasad-Sommerfeld or BPS states

$$\text{mass} = vg = \frac{v\, 2\pi n}{e}$$

but mass of broken gauge field ~ $ev \sim M_W$.

$$M_{mono} = M_W/\alpha > M_W \text{ typically.}$$

aside can also construct electrically charged magnetic monopole - dyon by taking $A_0^b = (x^b/r) f(r)$

Where does this come from?

Consider $\epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}$

$$= \epsilon_{\mu\nu\rho\lambda}\,\partial_\mu \underset{\wedge}{\overset{\mathrm{Tr}}{}}\left(A_\nu F_{\rho\lambda} - \tfrac{1}{3}A_\nu A_\rho A_\lambda\right)$$

~~[illegible]~~

<u>integrate</u>:

$$\int_M \epsilon_{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} = \int_{\partial M} d^3x\; \epsilon_{\cancel{\mu}\nu\rho\lambda} \underset{\wedge}{\overset{\mathrm{Tr}}{}}\left(A_\nu F_{\rho\lambda} - \tfrac{1}{3}A_\nu A_\rho A_\lambda\right)$$

but $F=0$ at $|x|=\infty$ and $A_\mu = g\partial_\mu g^\dagger$

$$\hookrightarrow \int F\tilde{F} \Rightarrow \int \epsilon_{ijk}\,\mathrm{Tr}(g\partial_i g^\dagger)(g\partial_j g^\dagger)(g\partial_k g^\dagger)$$

$= n$ up to constants

In fact

$$n = \frac{g^2}{16\pi^2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

Instantons

Again look for finite action configs which satisfy EOM. (Euclidean space)

$$S_{YM} = \int d^4x \frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu})$$

at infinity $|x| = \infty$ $F_{\mu\nu}$ must vanish faster than $1/|x|^2$ ie $\equiv$ to $\underset{A}{A_\mu} = 0$

$\rightarrow$ gauge potential must be pure gauge

$$A \Rightarrow g\, dg^\dagger \leftarrow \text{instantons as } x \to \infty$$

for some fixed g

take SU(2) as example

take $g = I x_4 + i x \cdot \sigma$ $\quad gg^\dagger = 1$

$\det g = 1 \rightarrow x_4^2 + \vec{x}^2 = 1$

ie group manifold is S^3

Map $\underline{S^3 \to S^3}$

$$n = -\frac{1}{24\pi^2} \int d^3x\, \epsilon^{ijk} \mathrm{Tr}((g\partial_i g^\dagger)(g\partial_j g^\dagger)(g\partial_k g^\dagger))$$

BPS bound on action

Consider $\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}\pm\tilde{F}_{\mu\nu}\right)^2$

$$= \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) \pm \mathrm{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$$

$$\therefore \int \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) \geq \left|\int \mathrm{Tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})\right|$$

$$\boxed{\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F_{\rho\lambda}} \qquad \uparrow \; = \frac{16\pi^2|n|}{g^2}$$

thus $S \geqslant 8\pi^2|n|/g^2$

minimum value of (Euclidean) action for solution to EOM ~~w~~ that interpolates between vacuum solution at $x_4=-\infty$ & ~~[illegible]~~ a different vacuum at $x_4=+\infty$ with

$n_+ - n_- = n$

Instanton is solution of $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$

1st order eq's! easier to solve.