hw5 – solutions (1)

(1) $\varphi(x,t) = v \tanh\left(\frac{\gamma m}{2}(x - x_0 - \beta t)\right)$

Lorentz boosted soln:

EOM: $\Box\varphi + V'(\varphi) = 0$ — set $\frac{M}{2} = 1$, $x_0 = 0$ } simplicity

notice $\frac{\partial^2}{\partial t^2} = \beta^2 \, \partial^2/\partial x^2$

$= \beta^2 \gamma^2 \partial^2/\partial z^2$

where $z = \gamma(x - \beta t)$

$\therefore \Box\varphi = (\partial^2/\partial t^2 - \partial^2/\partial x^2)\varphi = -\varphi''(z)$

but $\varphi''(z) + V'(\varphi) = 0$ — kink at rest

$\therefore$ boosted soln also satisfies EOM.

$p = \int T^{01}\, dx$ where $T^{01} = \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}$

$\hookrightarrow \therefore p = -\frac{v^2\gamma^2 m^2 \beta}{4} \int_{-\infty}^{\infty} \operatorname{sech}^4 \alpha \, d\alpha \, \frac{2}{\gamma m}$

do integral

$\hookrightarrow$ $\underline{p = -\gamma\beta M}$, $M = \frac{2}{3} m^3/\lambda$

Energy is similar.

$$E = \int dx \left(\tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}\varphi'^2 + V \right)$$

$$= \int dx \left(\frac{v^2 \gamma^2 m^2}{8} \operatorname{sech}^4(\alpha)(\beta^2+1) \right.$$ $\quad \alpha = \gamma(x-\beta t)$

$$\left. + \frac{\lambda}{4} v^4 \operatorname{sech}^4(\alpha) \right)$$

Simplifies $\longrightarrow$

$$\int_{-\infty}^{\infty} dx \left[\frac{m^2 v^2}{8} \operatorname{sech}^4(\alpha) \frac{2}{1-\beta^2} \right]$$

$= \gamma M$ using previous integral

or $E = \gamma M$

(2)

$$n = \int d^3x\, \epsilon^{ijk}\, Tr((g\partial_i g^\dagger)(g\partial_j g^\dagger)(g\partial_k g^\dagger))$$

$$\delta n = 3\int d^3x\, \epsilon^{ijk}\, Tr(\delta(g\partial_i g^\dagger)(g\partial_j g^\dagger)(g\partial_k g^\dagger))$$

$$gg^\dagger = 1 \rightarrow$$

$$\delta g^\dagger = -g^\dagger \delta g g^\dagger$$

$$\therefore \delta(g\partial_i g^\dagger) = \delta g \partial_i g^\dagger + g\partial_i \delta g^\dagger$$

$$= \delta g \partial_i g^\dagger - g\partial_i (g^\dagger \delta g g^\dagger)$$

$$= \cancel{\delta g \partial_i g^\dagger} - g(\partial_i g^\dagger)\delta g g^\dagger$$

$$- \partial_i \delta g\, g^\dagger$$

$$- \cancel{\delta g \partial_i g^\dagger}$$

$$\therefore \delta(g\partial_i g^\dagger) = -g\partial_i(g^\dagger\delta g)g^\dagger$$

$$\therefore \delta n = 3\int d^3x\, \epsilon^{ijk}\, Tr(\overbrace{-g^\dagger(g\partial_j g^\dagger)(g\partial_k g^\dagger)g}^{*}\ \partial_i(g^\dagger\delta g))$$

$*$ symmetric under exchange j, k

(note: $g\partial_j g^\dagger$ $\propto$ unit matrix if δg <u>infinitesimal</u>.)

$\therefore$ contract with $\epsilon_{ijk} \rightarrow 0$

$\therefore$ <u>$\delta n = 0$</u>

final part: if trans from coordinates ϵ^{ijk} and d^3x pick up Jacobian factors which cancel. Integrand is manifestly coordinate independent — topological

(3)

$$\langle\theta'|H|\theta\rangle = \sum_{m,n} e^{i(n\theta - m\theta')}\, e^{-S_0|m-n|}$$

$$= \sum_n e^{in(\theta-\theta')} + \sum_n \Big(\sum_{m<n} e^{i(n\theta - m\theta')}\, e^{-S_0|m-n|} + \sum_{m>n} e^{i(n\theta - m\theta')}\, e^{-S_0|m-n|} \Big)$$

let $p = n-m$ in 1st term
or $p = m-n$ in 2nd.

$$\hookrightarrow \quad 2\pi\delta(\theta-\theta') + \sum_n \Big(\sum_p e^{in(\theta-\theta') + ip\theta' - S_0 p} + \sum_p e^{in(\theta-\theta') - ip\theta' - S_0 p} \Big)$$

$$= 2\pi\delta(\theta-\theta') + 2\pi\delta(\theta-\theta') \sum_p e^{-S_0 p}\, 2\cos p\theta'$$

H diagonal in this basis ie eigenfunctions of H are "θ vacua"

keeping only $p=1 \rightarrow \underline{E \sim 2\cos\theta\, e^{-S_0} + 1}$

Alternatively

Consider $\langle m|H|\theta\rangle$

$$= \sum_n e^{in\theta} \langle m|H|n\rangle$$

$$= \sum e^{in\theta} e^{-|m-n|S_0}$$

$\tilde{m} = m-n$

$$\langle m|H|\theta\rangle = \sum_{\tilde{m}} e^{i(m-\tilde{m})\theta} f(\tilde{m})$$

note $H|\theta\rangle = \sum_m |m\rangle\langle m|H|\theta\rangle$

$$= \sum_m e^{im\theta} \sum_{\tilde{m}} e^{-i\tilde{m}\theta} f(\tilde{m}) |m\rangle$$

but $\sum_m e^{im\theta} |m\rangle = |\theta\rangle$

$$\therefore H|\theta\rangle = \sum_{\tilde{m}} e^{-i\tilde{m}\theta} f(\tilde{m}) |\theta\rangle$$

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