Lattice Gauge Theory

Seen that QCD (many non-abelian G.T) possess property called asymptotic freedom

Conversely α_s grows in I.R (confinement)
renders perturbation theory useless

* Perturbative QCD is also very ugly: gauge symmetry completely obscured in perturbative approach, proliferation of Feynman diagrams above 1 loop ....

* Way out? Wilson developed new approach. Key idea: keep exact gauge invariance (needed for unitarity) but do violence to Lorentz invariance

→ Put theory on lattice in spacetime

Namely this ruins gauge invariance

Derivatives (presumably) get replaced by finite difference operators like for PDEs

→ subtracting fields at different pts ← difference depends on gauge transformations!

Key is to construct an object - Wilson line - which ties 2 locations in spacetime together & transforms non-locally under GTs

$$W(x,y) \to e^{i\alpha(x)} W(x,y) e^{-\alpha(y)}$$

(think U(1) moment)

Then

$$W(x,y)\phi(y) - \phi(x)$$

$$\to e^{i\alpha(x)} W(x,y) e^{-i\alpha(y)} e^{i\alpha(y)} \phi(y) - e^{i\alpha(x)}\phi(x)$$

$$= e^{i\alpha(x)}\left[W(x,y)\phi(y) - \phi(x)\right] \;!$$

c.f $D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$ cont.

Allows us to construct gauge invariant

kinetic terms

Indeed $D_\mu \phi$ can now be defined as

$$\lim_{\delta x_\mu \to 0} \frac{W(x, x+\delta x)\phi(x+\delta x) - \phi(x)}{\delta x_\mu}$$

want $W(x,x) = 1$

thus $W(x, ~~y~~ x+\delta x) = 1 - ie\delta x^\mu A_\mu + \ldots$

$\to$ recover continuum ~~[illegible]~~ defn of

$$D_\mu = \partial_\mu - ieA_\mu$$

W for finite x, y

$$W(x,y) = \exp\left(ie \int_x^y A_\mu dx^\mu \right)$$

← abelian theory for simplicity..

+ correct to 1st order $y = x + \delta x$

Under gauge transformation

$$W(x,y) \to \exp\left(ie \int_x^y A_\mu dx^\mu + i \int_x^y \partial_\mu \alpha \, dx^\mu \right)$$

$$= e^{i\alpha(y)} \, W(x,y) \, e^{-i\alpha(x)}$$ QED

O.K now ready to go to lattice

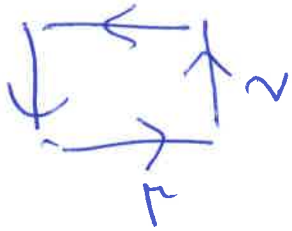
lattice sites x, links $x \to x + \hat{\mu}$

$$W \to e^{ieA_\mu(x \to x+\mu) a^\mu} = U_\mu(x)$$

i.e. gauge fields represented by ~~phases~~ phases on links of lattice

Notice

$P_{\mu\nu}$



gauge invariant

Wilson loop

$$P_{\mu\nu} = U_\mu(x) \, U_\nu(x+\mu) \, U^\dagger_\mu(x+\nu) \, U^\dagger_\nu(x)$$

guess (correctly!) ~~that~~ it is proportional to $F_{\mu\nu}^2$ for $a \to 0$

centre

$$W^{loop} = \exp\left(ie \oint A_\mu dx^\mu\right)$$

$$= \exp\left(i\frac{e}{2}\int_\Sigma F_{\mu\nu} d\sigma^{\mu\nu}\right)$$

Stokes

Non-abelian case

Few wrinkles

$$W(x,y) = \mathcal{P}\exp\left(ig\int A_\mu^a T^a dx^\mu\right)$$

path ordering

$$U_\mu(x) \to e^{iA_\mu^a T^a a}$$ lattice

valued in group

$$\sum_x \mathrm{Re}[\mathrm{Tr}\, P_{\mu\nu}] \to \int d^4x\, \mathrm{Tr}(F_{\mu\nu}^2)$$

$$U_\mu(x) \longrightarrow G(x)\, U_\mu(x)\, G^\dagger(x)$$ and lattice GT

$$Z_{latt} = \int DU \, e^{-\sum \mathrm{Re\,Tr}\, P_{\mu\nu}}$$

Since U is in group no need to gauge fix

(volume of group finite → integrating over gauge copies does not result in infinities)

Bonus

Finite # d.o.f. can probe theory numerically using simulations!

Bonus 2

New ~~...~~ formulation allows for strong coupling calculations

$$S_L = \frac{1}{g^2} \sum \mathrm{Re\,Tr}\, P_{\mu\nu}$$

expand in $1/g^2$

$$W^{loop}(R,T) \sim e^{-\sigma(RT)}$$

order parameter for confinement

$\sigma \sim$ string tension

to leading order in $1/g^2$ see non-zero σ

Why is this not "proof" of confinement.?

Because large violations of Lorentz invariance

due to lattice spacing effects

Need to show that the confining phase

of lattice QCD survives cont. limit

$\to$ lot of evidence but no analytic proof.

Fermions

While gauge fields map smoothly onto discrete spacetime the same can not be said for fermions

$A_\mu(x)$ – vector $\longrightarrow$ links on lattice

ψ – spinor $\longrightarrow$? what geometrical object on discrete spacetime ?

Go ahead & try ...

put $\psi(x)$ on sites

$$S = \sum_x \bar{\psi}(x)\gamma_\mu\left[U_\mu(x)\psi(x+\mu) - U_\mu^\dagger(x-\mu)\psi(x-\mu)\right]$$

$\psi(x) \to G(x)\psi(x)$

$U_\mu(x) \to G(x)\, U_\mu\, G^\dagger(x+\mu)$

Gauge invariant

~~cont~~ naive cont: limit looks o.k.

But problems lurk...

Put $U = I$ for now.

Go to (lattice) momentum space.

$$\sum_p \bar{\psi}(p)\,\gamma_\mu \left(\frac{e^{ip_\mu a} - e^{-ip_\mu a}}{2}\right)\psi(-p)$$

lattice propagator:

$$\frac{1}{\gamma_\mu \sin p_\mu a + m}$$

<u>$a \to 0$</u> pole at $i\not{p}\,a + m \simeq 0$

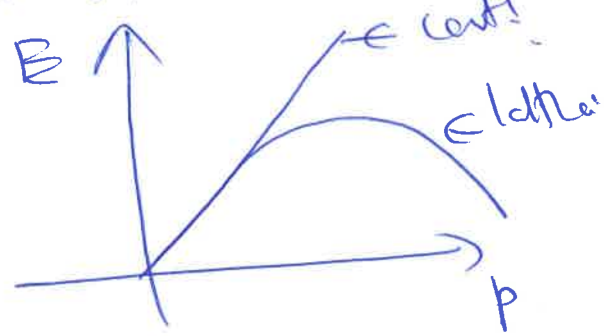
(like cont^m prop in Euclidean space)

But note many other poles too!

$$\sin p_\mu a \simeq 0 \qquad p_\mu = (\pi/a, 0, 0, 0)$$
$$= (0, \pi/a, 0, 0)$$

$\Rightarrow$ <u>2^D poles</u> in D dimensions

$\boxed{\text{Fermion doubling}}$



Topological thms guarantee this problem cannot be fixed by using other forms of discrete derivative unless you break chiral symmetry explicitly

(Wilson)

To talk about chiral fermions need to be able to decouple L & R states.

- not possible

chiral gauge theories cannot be achieved using Wilson fermions

(overlap / domain wall fermions have better chiral properties but still fail to realize chiral gauge theory)

One partial way around this

Kähler-Dirac fermions

Contn

$$(\gamma^\mu \partial_\mu + m)\psi^i = 0 \quad (*)$$

$i = 1..4$

4 degenerate copies
SO(4) flavor

Regard ψ^i as 4×4 matrix under new rotation group

$$\mathrm{diag}(\underset{\text{rotations}}{SO(4)} \times \underset{\text{flavor}}{SO(4)})$$

$$\therefore \Psi = \eta I + \psi_\mu \gamma_\mu + \chi_{\mu\nu}\gamma_\mu\gamma_\nu + \theta_\mu \gamma_5 \gamma_\mu + K\gamma_5 .$$

a scramble p-forms $\Lambda = (\eta, \psi, \chi, \theta, K)$ into KD field

Dirac eqn $(*)$ rewritten

$$(d - d^\dagger)\Lambda + m\Lambda = 0$$

(note $(d-d^\dagger)^2 = dd^\dagger - d^\dagger d = \Box$ a la Dirac)

Advantages

(1) Know how put P forms on lattice (associated with elementary p-cells

$p=0$ sites
$p=1$ links
$p=2$ plaquettes
. . .

(2) Can discretize KD eqn without getting additional doublers (topology again)

(3) Can map into popular method for lattice QCD – staggered fermions.

But contm theory still contains eql #s of L & R fields
(4 Dirac or $m=0$ 2 Dirac)