Gravity as Gauge Theory

In GR learn that fundamental object is

$g_{\mu\nu}$ — metric (think about Euclidean space simplicity)

symmetric tensor

First step: write $g_{\mu\nu} = e_\mu^a e_\nu^b \delta^{ab}$ $\quad a,b = 1..4$.

e_μ^a: vielbein | frame field

You can always do this

$g_{\mu\nu} \sim 10$ dof $\quad e_\mu^a \to \underline{16}$

in fact you have too many dof! with e.

O.K. clear that $g_{\mu\nu}$ invariant under orthogonal rotations of e_μ^a at each x

$$e'^a_\mu = O^{ab}(x) e^b_\mu(x)$$

$$OO^T = I$$

e_μ set of vectors at each pt in spacetime. ~~[illegible]~~

e_μ^a ~~[illegible]~~ space components in tangent space

Imagine parallel transporting vector e_μ. Components relative to local tangent space will change

Since length unchanged - this will look like a <u>rotation</u>

$$\delta e^a_\mu = \partial_\nu e^a_\mu(x)\, dx^\nu = -\omega_\nu^{ab}(x)\, e^b_\mu(x)$$

antisymmetric in a,b

Suppose I rotate the frame $e \to e'$ what happens to ω ?

$$\delta e'^a_\mu(x) = \partial_\nu\left(O^{ab} e^b{}_\mu(x)\right) dx^\nu$$

$$= \left(\partial_\nu O^{ab}\, e^b_\mu + O^{ab}\, \partial_\nu e^b_\mu\right) dx^\nu$$

$$= \partial_\nu O^{ab}\, O^{T\,bc}\, e'^c_\mu(x) + O^{ab}\left(-\omega_\nu^{bc}\right) e^c_\mu$$

$$= \left[(\partial_\nu O^{ab})\, O^{T\,bc} - O^{ab}\,\omega_\nu^{bd}\, O^{T\,dc}\right] e'^c_\mu$$

$$e\, \omega'^{\,ac}_{\nu} = (\partial_\nu O^{ab}) O^{T\,bc} - O^{ab}\, \omega_\nu^{\,bd}\, O^{T\,dc}$$

transforms like gauge field!

ω - spin connection — historical reasons for name. — is there to make sure physics does not depend on choice of frame e_μ. — mops up extra 6 dof which are not determined by $g_{\mu\nu}$

O.K, keep going. Given a connection / gauge field know how to compute curvature / field strength

$$R_{\mu\nu} = [D_\mu, D_\nu] = \partial_{[\mu} \omega_{\nu]} + [\omega_\mu, \omega_\nu]$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu + \omega_\mu^{ab}$$

^ covariant derivative

gauge group

is $SO(4)$ [$SO(3,1)$ Minkowski space]

ie local Lorentz transformations

As expected R transforms ④
covariantly

$$R^{ab}_{\mu\nu} \rightarrow O^{ac} R_{\mu\nu} O^{T\,cb}$$ under
local
SO(4) rotations

We are now in position to
reconstruct Einstein Hilbert gravity in terms
of ω_μ, e_μ (Palatini formulation)

① $g_{\mu\nu}$ is output — components of e's. Cannot
use it to contract indices. Good ← ensure
~~that~~ formulation is background independent

② Have e_μ, $R_{\mu\nu}$. SO(4) gauge invariance
precludes adding ω directly - -

Very small set of possible terms - - -

Note : Inclusion of fermions requires use of
vierbein formalism since spinors only defined
relative to local Lorentz frame

$$\int R^{ab}_{\mu\nu}\, e^c_\rho\, e^d_\lambda\, \epsilon_{\mu\nu\rho\lambda}\, \epsilon_{abcd} \qquad \text{(A)}$$

$$\int e^a_\mu\, e^b_\nu\, e^c_\rho\, e^d_\lambda\, \epsilon_{\mu\nu\rho\lambda}\, \epsilon_{abcd} \qquad \text{(B)}$$

note integral of 4-form is automatically coordinate indep

also $\times \int (D^{ab}_\mu e^b_\nu)\, R^{cd}_{\rho\lambda}\, \epsilon_{\mu\nu\rho\lambda}$

cannot satisfy indices...

etc etc

notice (B) $\equiv \int \det e$ ← matrix

but $\det e = \sqrt{\det g}$!

ie cosmological constant term!

What about (A) ?

notice: $\int \cancel{Tr}(\epsilon_{\mu\nu\rho\lambda} R^{ab}_{\mu\nu} R^{cd}_{\rho\lambda})\epsilon^{abcd}$ topological invariant...

ignore

Contraction with $e_\mu{}^a$ or its (matrix) inverse can raise/lower Greek/roman indices corresponding to spacetime and tangent space.

eg $\quad R^{ab}{}_{\mu\nu}\, e^\rho{}_a\, e^\lambda{}_b = R^{\rho\lambda}{}_{\mu\nu}$

inverse e

carries only 4 spacetime indices

Guess this is the Riemann curvature.

correct! but notice first that the

torsion free condition $\partial_{[\mu} e^a_{\nu]} + \{\omega^{ab}_{[\mu}, e^b_{\nu]}\} = 0$

means $\omega = \omega(e)$

$\therefore\ R^{ab}{}_{\mu\nu}(\omega) = R^{ab}_{\mu\nu}(e) = R^{ab}_{\mu\nu}(g)$

This is necessary in order for

$R^{\rho\lambda}_{\mu\nu}(\omega) =$ Riemann curvature.

which is function of $g_{\mu\nu}$

(6.5)

Action can be written

$$S = \int \det e \; e^{\rho}_{a} e^{\lambda}_{b} R^{ab}_{\ \ \rho\lambda}$$

$\uparrow$

$\sqrt{\det g}$

Notes

vary $e^{\rho}_{a} \rightarrow e^{\lambda}_{b} R^{ab}_{\ \ \rho\lambda} \equiv R^{a}_{\ \rho} = 0$

ie $R_{\mu\nu} = 0$

vary $\omega^{ab}_{\ \ \mu}$

$\rightarrow \quad D_{\mu} e^{\lambda}_{\ \nu} = 0$

$\uparrow$

torsion condition

Actually ordinary GR can be written in 1st order form when Christoffel connection $\Gamma^{\mu}_{\nu\lambda}$ is treated as independent of metric $\leftarrow$ Palatini formulation.

By varying both obtain same EOM.

The vierbein / spin connection formalism (ω, e) most neatly maps onto (G, Γ) Palatini

(ω, e) formalism (Cartan)

Almost like Yang-Mills but contains e .. any closer connection?

Consider gauge group $SO(5)$ rather than $SO(4)$

Generators $SO(5)$: (8)

$$T^{AB} \cancel{}, \quad A, B = 1 \ldots 5.$$

Split into 2 sets

$$T^{AB} \to T^{ab} \oplus T^{a5}$$

$\uparrow$ $a, b = 1 \ldots 4.$

Generators of $SO(4)$

Similarly $SO(5)$ curvature decomposes as

$$F^{AB} = \tilde{R}^{ab} + \tilde{M}^{5b} \qquad \text{when.} \quad \tilde{R} = R + \frac{1}{\ell^2}[e, e]$$

$$A^{AB} \to \omega^{ab} + \frac{1}{\ell} e^{5b} \qquad \tilde{M} = De.$$

$\leftarrow$ length scale...

Break $SO(5)$ explicitly with action

(SO(4) remains)

$$S = \int d^4x \, \epsilon_{\mu\nu\rho\lambda} \, \epsilon^{ABCD5} \left(F^{AB} F^{CD} \right)$$

$$= \int d^4x \, \epsilon_{\mu\nu\rho\lambda} \, \epsilon^{abcd} \left((\tilde{R} + \tilde{M})^{ab} (\tilde{R} + \tilde{M})^{cd} \right)$$

$$= \int d^4x \, \epsilon_{\mu\nu\rho\lambda} \, \epsilon^{abcd} \Big(2\tilde{R}^{ab} \cancel{\tilde{M}^{cd}} + \cancel{\tilde{M}^{ab}} \tilde{M}^{cd} + \tilde{R}^{ab} \tilde{R}^{cd} \Big)$$

Thus EH gravity $(+\Lambda)$

"almost" Yang-Mills theory. Requires $\underline{SO(5)}$

(de Sitter $SO(4,1)$)

gauge symmetry explicitly broken to $SO(4)$

($SO(3,1)$)

in action.

This construction works in any dimension.

In odd dimensions it gets more beautiful.

Consider 3D

Natural dimensionless action is Chern-Simons

Analog of $SO(5) \to SO(4)$ Euclidean space

$$S = \int \epsilon_{\mu\nu\rho} A_\mu^{ab} F_{\nu\rho}^{cd} \epsilon^{abcd}$$

where

$$A_\mu^{ab} = \omega_\mu^{ab} + \frac{1}{\ell} e_\mu^{4a}$$

$$S = \int \epsilon_{\mu\nu\rho} \left(\omega_\mu^{ab} + \frac{1}{\ell} e_\mu^{4a}\right)\left[R_{\nu\rho}^{cd} + \frac{e_\nu^{4c} e_\rho^{4d}}{\ell^2}\right] \epsilon^{abcd}$$

$$= \int \epsilon_{\mu\nu\rho} \left(\frac{e_\mu^{4a} e_\nu^{4b} e_\rho^{4c}}{\ell^3} + \frac{1}{\ell} e_\mu^{4a} R_{\nu\rho}^{cd} \right)$$